Minimal time optimal control problems

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Summary

Introduction

- Optimal Control Problem

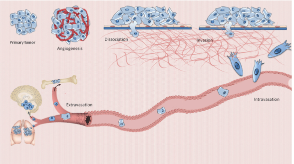
$$\min J(f) \qquad \left\{ \begin{array}{l} a(u) = B(f) \text{ in } \Omega \times (0,T) \\ + \text{ bounded conditions} \end{array} \right. \tag{1}$$

- Minimum $\Rightarrow J'(f) = 0.$
- Minimal Time Optimal Control Problem

$$\min J(T, f) \qquad \left\{ \begin{array}{l} a(u) = B(f) \text{ in } \Omega \times (0,T) \\ + \text{ bounded conditions} \end{array} \right. \tag{2}$$

Motivation

Medicine



Biology, Ecology

Economy

Engineering

Introduction

Linear ODE

$$\begin{cases} y_t + ay = h(t), \quad t \in (0,T), \\ y(0) = y_0, \end{cases} \tag{3}$$

Nonlinear ODE

$$\begin{cases} y_t + H(y) = h(t), \quad t \in (0,T), \\ y(0) = y_0, \end{cases} \tag{4}$$

$$\begin{cases} H : \mathbb{R} \mapsto \mathbb{R} \text{ is of class } \mathcal{C}^1, \\ 0 \leq H'(s) \leq C \quad \forall s \in \mathbb{R}. \end{cases} \tag{5}$$

Introduction

Heat equation

$$\begin{cases} \theta_t - \Delta\theta = h 1_\omega, & (x,t) \in Q_T := \Omega \times (0,T), \\ \theta = 0, & (x,t) \in \Sigma_T := \partial\Omega \times (0,T), \\ \theta(0) = \theta_0, \end{cases} \tag{6}$$

Problem

$$\begin{cases} \text{Minimize } \phi(T,h) := \dfrac{T^2}{2} + \dfrac{b}{2} \displaystyle\iint_{\omega\times(0,+\infty)} |h|^2 \, dx \, dt, \\ \text{Subject to: } (T,h) \in \mathbb{R}_+ \times L^2(\omega \times (0,+\infty)), \\ \qquad (\theta,h) \text{ solves (6)}, \\ \qquad \|\theta(T) - \theta_d\| = \delta, \end{cases} \tag{7}$$

Summary

Linear ODE

$$\mathcal{H}_{ad} := \{(T,h):\ T \geq 0,\ h \in L^2(0,+\infty),\ |y_h(T) - y_d| = \delta\},$$

Theorem

Let $a, \delta > 0$, $b > 0$ *and* $y_0, y_d \in \mathbb{R}$ *with* $|y_0 - y_d| > \delta$. *Then, there exists a unique solution* $(T,h) \in \mathcal{H}_{ad}$ *to the:*

$$\begin{cases} \textit{Minimize } \phi(T,h) := \dfrac{T^2}{2} + \dfrac{b}{2}\displaystyle\int_0^{+\infty} |h|^2\, dt, \\ \textit{Subject to: } (T,h) \in \mathbb{R}_+ \times L^2(0,+\infty), \\ \qquad\qquad\quad (y,h) \textit{ solves } (3), \\ \qquad\qquad\quad |y(T) - y_d| = \delta, \end{cases} \tag{8}$$

Linear ODE

Theorem

Let $(T, h) \in \mathcal{H}_{ad}$ *solution to the* (8)*, then there exists* $\lambda > 0$ *and* $\psi = \psi(t)$ *such that:*

$$\begin{cases} y_t + ay = h, \quad t \in (0,T), \quad y(0) = y_0, \\ -\psi_t + a\psi = 0, \quad t \in (0,T), \\ \psi(T) = y(T) - y_d, \quad |y(T) - y_d| = \delta, \\ h = -\dfrac{1}{\lambda b}\psi \quad \text{in } (0,T), \quad T = -\dfrac{1}{\lambda}(y(T) - y_d) y_t(T). \end{cases} \tag{9}$$

Nonlinear ODE

$$\mathcal{H}_{ad} := \{(T,h) : \ T \geq 0, \ h \in L^2(0,+\infty), \ |y_h(T) - y_d| = \delta\},$$

Theorem

There exists at least one solution $(T,h) \in \mathcal{H}_{ad}$.

Theorem

Let $(T,h) \in \mathcal{H}_{ad}$ *be a solution* $\Rightarrow \exists \ \lambda > 0$ *and* $\psi = \psi(t)$ *such that:*

$$\begin{cases} y_t + H(y) = h, \quad t \in (0,T), \quad y(0) = y_0, \\ -\psi_t + H'(y)\psi = 0, \quad t \in (0,T), \\ \psi(T) = y(T) - y_d, \quad |y(T) - y_d| = \delta, \\ h = -\dfrac{1}{\lambda b}\psi \quad \text{in } (0,T), \quad T = -\dfrac{1}{\lambda}(y(T) - y_d)y_t(T). \end{cases} \tag{10}$$

Heat Equation

Theorem

There exists at least one solution $(T, h) \in \mathcal{H}_{ad}$.
If $(T, h) \in \mathcal{H}_{ad}$ *be a solution* $\Rightarrow \exists\ \lambda > 0$ *and* $\psi = \psi(x,t)$ *such that:*

$$\begin{cases} \theta_t - \Delta\theta = h1_\omega, \quad (x,t) \in Q_T, \\ \theta = 0, \quad (x,t) \in \Sigma_T, \quad \theta(0) = \theta_0, \\ -\psi_t - \Delta\psi = 0, \quad (x,t) \in Q_T, \\ \psi = 0, \quad (x,t) \in \Sigma, \quad \psi(T) = \theta(T) - \theta_d, \\ \|\theta(T) - \theta_d\| = \delta, \\ h = -\dfrac{1}{\lambda b}\, \psi|_{\omega\times(0,T)}, \\ T = -\dfrac{1}{\lambda}\Big((\theta(T) - \theta_d), \theta_t(T)\Big). \end{cases} \tag{11}$$

Summary

Algorithms

- Fixed Point Method (Linear ODE)
- Direct Method
  1. For some T^k and $\mu^j = 1/\lambda^j$ compute

     $$\|\theta^{k,j}(T^k) - \theta_d\| - \delta \quad \text{and} \quad -\mu^j(\theta^{k,j}(T) - \theta_d, \theta_t^{k,j}(T^k)),$$

  2. Compute a solution (T_*, μ_*) to the system

     $$T = -\mu(\theta(T) - \theta_d, \theta_t(T)), \quad \|\theta(T) - \theta_d\| = \delta \tag{12}$$

- Penalty

$$\tilde{\phi}(T, h; \mu^n) := \frac{T^2}{2} + \frac{b}{2} \iint_{\omega\times(0,+\infty)} |h|^2 + \frac{1}{2\mu^n}\Big((\|\overline{\theta}(T) - \theta_d\| - \delta)^2 + T_-^2\Big).$$

- Augmented Lagrangian

Linear ODE

$$\begin{cases} y_t + ay = h, \quad t \in (0,T), \quad y(0) = y_0, \\ -\psi_t + a\psi = 0, \quad t \in (0,T), \\ \psi(T) = y(T) - y_d, \quad |y(T) - y_d| = \delta, \\ h = -\dfrac{1}{\lambda b}\psi \quad \text{in } (0,T), \quad T = -\dfrac{1}{\lambda}(y(T) - y_d)y_t(T). \end{cases} \tag{13}$$

Parameters

$$\delta = 10^{-5}, \quad y_d = 10, \quad b = 0.5 \text{ and } a = 0.5.$$

Linear ODE

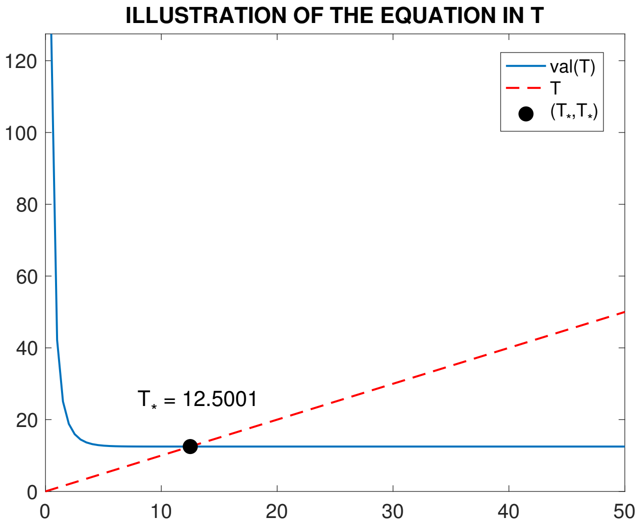
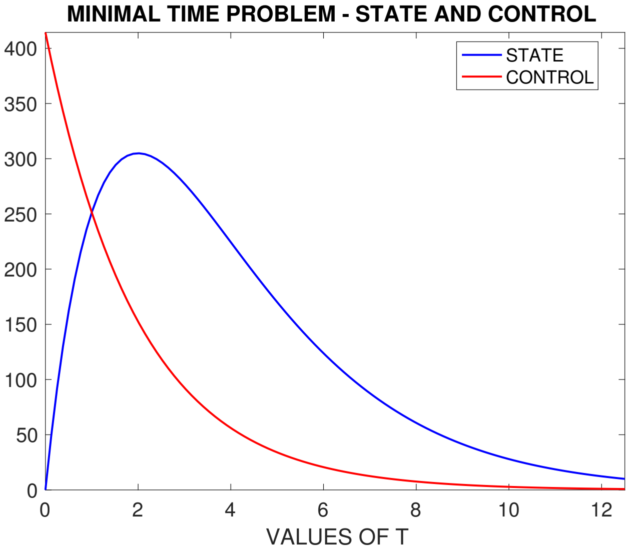


ILLUSTRATION OF THE EQUATION IN T
val(T)
T
(T_*,T_*)
$T_* = 12.5001$
0
20
40
60
80
100
120
0
10
20
30
40
50

Linear ODE


MINIMAL TIME PROBLEM - STATE AND CONTROL
STATE
CONTROL
400
350
300
250
200
150
100
50
0
0
2
4
6
8
10
12
VALUES OF T

Heat equation

Direct Method

1. For some T^k and $\mu^j = 1/\lambda^j$ compute

$$\|\theta^{k,j}(T^k) - \theta_d\| - \delta \quad \text{and} \quad -\mu^j(\theta^{k,j}(T) - \theta_d, \theta_t^{k,j}(T^k)),$$

2. Compute a solution (T_*, μ_*) to the system

$$T = -\mu(\theta(T) - \theta_d, \theta_t(T)), \quad \|\theta(T) - \theta_d\| = \delta \tag{14}$$

Parameters

$$\delta = 0.07, \quad b = 100,$$

$$T_0 = 5, \quad T_1 = 20, \quad \Delta t = 1.25,$$

$$\mu_0 = 120, \quad \mu_1 = 160, \quad \Delta\mu = 10.$$

Heat equation

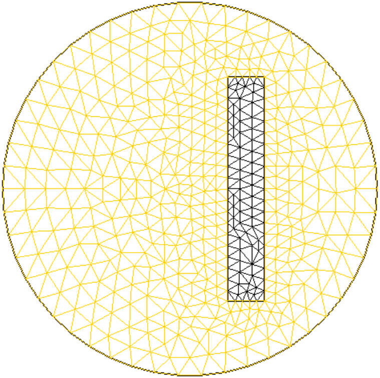


Figure: The mesh.

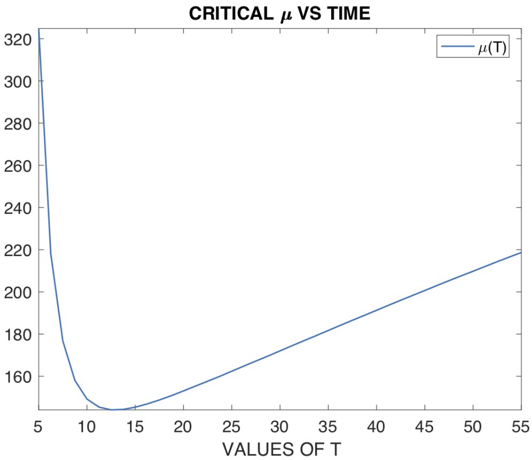


Figure: The values of μ that solve the equation $\|\theta(T) - \theta_d\| = \delta$.

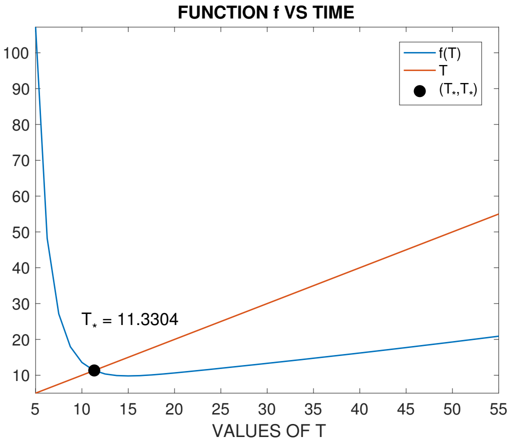


Figure: Minimal time T_*.

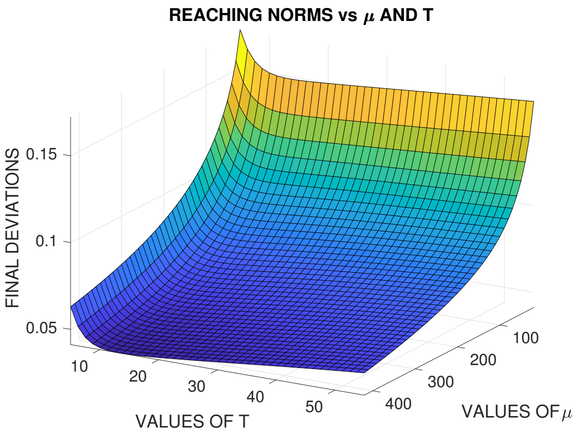


Figure: Values to the norm $\|\theta(T) - \theta_d\| - \delta$ versus μ and T.

Summary

Open Questions and Perspectives

- Uniqueness to the nonlinear ODE and heat problem?
- Convergence to the direct algorithm and to the Augmented Lagrangian algorithm? To develop.
- Other equations: semilinear/nonlinear PDEs, Stokes equation, Navier-Stokes equations, wave PDE...? To develop.

Contributions

- New vision of minimal control time problem $\rightarrow$ Optimal control problems.
- Existence and characterization results to linear and nonlinear ODE and heat equation.
- Characterization to the solution.
- Algorithms to compute the solution and numerical experiences.