Regularidad para soluciones de problemas de control con potenciales

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We consider $\Omega \subset \mathbb{R}^N$ a bounded open set. We are interested in the optimization problem of the form:

$$\min \int_\Omega \left[j(x,u) + \psi(m) \right] dx,$$

governed by the state equation

$$\begin{cases} -\Delta u + mu = f & \text{in } \Omega \\ u \in H_0^1(\Omega). \end{cases}$$

where, the control variable m is assumed to be nonnegative, $f \in L^2(\Omega)$, and j, ψ are suitable integrands.

If we assume that ψ has a superlinear growth, then $m \in L^1(\Omega)$.

Compliance case

We consider $j(x,u) = f(x)u$, the problem can be written in the variational form

$$\min \Big\{ -2\mathscr{E}(m) + \Psi(m) \ : \ m \in L^1(\Omega), \ m \geq 0 \Big\},$$

where

$$\Psi(m) = \int_\Omega \psi(m)\,dx$$

$$\mathscr{E}(m) = \min \left\{ \int_\Omega \Big[\frac{1}{2}|\nabla u|^2 + \frac{1}{2} m u^2 - f(x)u \Big]\,dx \ : \ u \in H_0^1(\Omega) \right\}.$$

The control variable m can be eliminated, obtaining the *auxiliary* variational problem

$$\min \left\{ \int_\Omega \Big[|\nabla u|^2 + \psi^*(u^2) - 2f(x)u \Big]\,dx \ : \ u \in H_0^1(\Omega) \right\}, \qquad (1)$$

where ψ^* denotes the Legendre-Fenchel conjugate of the function ψ.

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We put $g(s) = s(\psi^*)'(s^2)$.

The unique solution $\hat{u}$ of (1) can be obtained by the PDE

$$\begin{cases} -\Delta u + g(u) = f \quad \text{in } \Omega \\ u \in H_0^1(\Omega), \end{cases} \tag{2}$$

and the optimal control $\hat{m}$ can be then recovered as

$$\hat{m} = (\psi^*)'(\hat{u}^2).$$

For general $j(x, u)$ the above procedure does work.

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Aims

- Existence of optimal solutions.
- **Aditional regularity properties.**

$$\hat{m}\hat{u}\hat{z} \in BV(\Omega).$$

No more regularity for $\hat{m}$ (for general assumptions on ψ):

$$\psi(s) = \begin{cases} s & \text{if } s \in [\alpha, \beta] \\ +\infty & \text{otherwise,} \end{cases} \qquad (\text{with } 0 \leq \alpha < \beta)$$

the optimal control $\hat{m}$ is *bang-bang*:

$$\hat{m} = \alpha + (\beta - \alpha)1_E$$

for a suitable set E, with finite perimeter.

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We analice the existence and regularity properties of the solutions of a semi-linear problem

$$\begin{cases} -\Delta u + g(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

with g **non-decreasing** and **not necessarily continuous** and $f \in L^1(\Omega)$.

Definition

Assume $f \in L^1(\Omega)$. We say that a pair (u, w) is a solution of (3) if it satisfies

$$\begin{cases} u \in H_0^1(\Omega) & \text{if } N = 1 \\ u \in W_0^{1,p}(\Omega)\ \forall p < \dfrac{N}{N-1}, \quad T_k(u) \in H_0^1(\Omega)\ \forall k > 0 & \text{if } N \geq 2. \end{cases}$$

$$w \in L^1(\Omega), \quad g_-(u) \leq w \leq g_+(u) \ \text{ a.e. in } \Omega,$$

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Omega wv\, dx = \int_\Omega fv\, dx \\ \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega) \text{ such that } \exists k > 0 \ \text{ with } \nabla v = 0 \text{ a.e. in } \{|u| > k\}. \end{cases}$$

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Existence and uniqueness

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and $g : \mathbb{R} \to \mathbb{R}$ a non-decreasing function. Then, for every $f \in L^1(\Omega)$, there exists a unique solution (u, w) of (3) in the sense of the above Definition. Moreover, it satisfies

$$\|w - g_+(0)\|_{L^1(\Omega)} \leq \|f - g_+(0)\|_{L^1(\Omega)},$$

$$\begin{cases} \|u\|_{H_0^1(\Omega)} \leq C\|f\|_{L^1(\Omega)} & \text{if } N = 1, \\ \|T_k(u)\|^2_{H_0^1(\Omega)} \leq \displaystyle\int_\Omega |f - g_+(0)||T_k(u)|\,dx \quad \forall\, k > 0, & \\ \|u\|_{W_0^{1,p}(\Omega)} \leq C\|f - g_+(0)\|_{L^1(\Omega)} \quad \forall\, p < \dfrac{N}{N-1} & \text{if } N \geq 2. \end{cases}$$

In these estimates the constant C only depends on $|\Omega|$, p, and N.

Assuming f in $L^q(\Omega)$, with $q > 1$, from classical estimates for renormalized solutions and Stampacchia's estimates give:

$$\begin{cases} u \in W_0^{1,\frac{Nq}{N-q}}(\Omega) & \text{if } 1 < q \leq \dfrac{2N}{N+2} \\ u \in H_0^1(\Omega) \cap L^{\frac{Nq}{N-2q}}(\Omega) & \text{if } \dfrac{2N}{N+2} < q < \dfrac{N}{2} \\ u \in H_0^1(\Omega) \cap L^{\infty}(\Omega) & \text{if } \dfrac{N}{2} < q. \end{cases}$$

For $f \in L^{\frac{2N}{N+2}}(\Omega)$, we have $u \in H_0^1(\Omega)$.
In this case, in a simpler way it corresponds as the unique solution of the strictly convex minimum problem

$$\min_{v \in H_0^1(\Omega)} \int_\Omega \left(\frac{1}{2}|\nabla v|^2 + G(v) - fv \right) dx,$$

with $G : \mathbb{R} \to \mathbb{R}$ defined by

$$G(s) = \int_0^s g(r)\, dr \qquad \forall\, s \in \mathbb{R}.$$

Continuous dependence and maximum principle

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $g : \mathbb{R} \to \mathbb{R}$ a non-decreasing function. For $f_1, f_2 \in L^1(\Omega)$ we take (u_1, w_1), (u_2, w_2) solutions of (3) with $f = f_1$ and $f = f_2$ respectively. Then, we have

$$\|w_1 - w_2\|_{L^1(\Omega)} \leq \|f_1 - f_2\|_{L^1(\Omega)};$$

$$\|T_k(u_1 - u_2)\|_{H_0^1(\Omega)} \leq k\|f_1 - f_2\|_{L^1(\Omega)} \qquad \forall\, k > 0;$$

$$\begin{cases} \|u_1 - u_2\|_{H_0^1(\Omega)} \leq C\|f_1 - f_2\|_{L^1(\Omega)} & \text{if } N = 1 \\ \|u_1 - u_2\|_{W_0^{1,p}(\Omega)} \leq C\|f_1 - f_2\|_{L^1(\Omega)} \quad \forall\, p \in \left(1, \dfrac{N}{N-1}\right) & \text{if } N \geq 2. \end{cases}$$

The constant $C > 0$ in this last inequality only depends on $|\Omega|$ if $N = 1$. For $N \geq 2$ it only depends on N, p and $|\Omega|$. In addition,

$$f_1 \leq f_2 \text{ a.e. in } \Omega \implies u_1 \leq u_2 \text{ a.e. in } \Omega.$$

BV regularity

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class $C^{1,1}$, and $g : \mathbb{R} \to \mathbb{R}$ a non-decreasing function. Then, for $f \in BV(\Omega)$, the solution (u, w) of (3) *is in $W^{2,\frac{N}{N-1}}(\Omega) \times BV(\Omega)$. Moreover, there exists $C > 0$ depending only on Ω such that*

$$\|u\|_{W^{2,\frac{N}{N-1}}(\Omega)} \leq C\big(\|f\|_{BV(\Omega)} + |g_+(0)|\big),$$

$$\|w\|_{BV(\Omega)} \leq C\big(\|f\|_{BV(\Omega)} + |g_+(0)|\big).$$

Optimal potentials problems

$$\min \int_\Omega \big(j(x,u)+\psi(m)\big)\,dx$$
$$\begin{cases} -\Delta u + m\,u = f \ \text{ in } \Omega \\ u\in H^1_0(\Omega), \quad m\in L^1(\Omega), \quad m\geq 0 \ \text{ a.e. in } \Omega, \end{cases} \tag{4}$$

with ψ is a lower semicontinuous convex function. $m\geq 0$ in Ω can be introduced by

$$\psi(s)=+\infty \qquad \forall\, s<0.$$

State equation in (4), must be understood in the variational sense

$$\begin{cases} u\in H^1_0(\Omega)\cap L^2_m(\Omega) \\ \displaystyle\int_\Omega (\nabla u\cdot\nabla v + muv)\,dx = \langle f,v\rangle \quad \forall\, v\in H^1_0(\Omega)\cap L^2_m(\Omega). \end{cases}$$

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Theorem

Let $\Omega \subset \mathbb{R}^N$ *be a bounded open set,* $j : \Omega \times \mathbb{R} \to \mathbb{R}$ *measurable in the first component and lower semicontinuous in the second one, bounded by*

$$a(x) - c|s|^2 \leq j(x, s)$$

for suitable $c \geq 0$ *and* $a \in L^1(\Omega)$*, and* $\psi : \mathbb{R} \to [0, \infty]$ *a convex lower semicontinuous function with* $\mathrm{dom}(\psi) \subset [0, \infty)$*, such that*

$$\lim_{s \to \infty} \frac{\psi(s)}{s} = \infty.$$

Then, for every $f \in H^{-1}(\Omega)$*, problem* (4) *has a least one solution* $\hat{m} \in L^1(\Omega)$*.*

Optimality conditions

We assume:

$$f \in W^{-1,r}(\Omega), \quad \text{with} \begin{cases} r \geq 2 & \text{if } N = 1 \\ r > N & \text{if } N \geq 2. \end{cases} \tag{5}$$

and $j(x,.) \in C^1(\mathbb{R})$ such that

$$j(.,0) \in L^1(\Omega), \qquad \max_{|s| \leq M} |\partial_s j(\cdot, s)| \in L^{\frac{r}{2}}(\Omega) \quad \forall\, M > 0. \tag{6}$$

We put the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(\tau) = \max \left\{ s \in \text{dom}(\psi) \ : \ \tau \in \partial\psi(s) \right\}.$$

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Theorem

Assume that in the above Theorem, the right-hand side f and the function j satisfy (5) and (6) respectively. Then, if $\hat{m}$ is a solution of (4), $\hat{u}$ is the corresponding state function, solution of

$$\begin{cases} -\Delta \hat{u} + \hat{m}\,\hat{u} = f & \text{in } \Omega \\ \hat{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

and $\hat{z}$ is the adjoint state, solution of

$$\begin{cases} -\Delta \hat{z} + \hat{m}\,\hat{z} = \partial_s j(x, \hat{u}) & \text{in } \Omega \\ \hat{z} = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

we have

$$\hat{m} \in L^\infty(\Omega), \quad \hat{u}\hat{z} \in \partial\psi(\hat{m}), \quad h_-(\hat{u}\hat{z}) \leq \hat{m} \leq h(\hat{u}\hat{z}), \quad \text{a.e. in } \Omega.$$

Regularity of function h

Proposition

The function h satisfies

$$h \in C^0(\mathbb{R}) \iff \psi \text{ is strictly convex.}$$

$$h \in \mathrm{Lip}(\mathbb{R}) \iff 0 < \inf_{\substack{s_1, s_2 \in \mathrm{dom}(\psi) \\ s_1 < s_2}} \frac{d_-\psi(s_2) - d_+\psi(s_1)}{s_2 - s_1}.$$

Main result

Theorem

In addition to the conditions in the existence Theorem, we assume $\Omega \in C^{1,1}$,

$$g(\tau) := h(\tau)\tau \qquad \forall\, \tau \in \mathbb{R},$$

non-decreasing in τ, *and*

$$\max_{|s|\leq M} |\nabla_x \partial_s j(.,s)| \in L^q(\Omega), \quad \max_{|s|\leq M} |\partial^2_{ss} j(.,s)| \in L^1(\Omega), \quad \forall\, M > 0,$$

with

$$q \geq \frac{2N}{N+1} \text{ if } 1 \leq N \leq 2, \qquad q > \frac{N}{2} \text{ if } N \geq 3.$$

Then, for every $f \in BV(\Omega) \cap L^q(\Omega)$ *and every solution* $\hat{m}$ *of* (4), *we have*

$$\hat{u}, \hat{z} \in W^{2,q}(\Omega), \qquad \hat{m}\hat{u}\hat{z} \in BV(\Omega),$$

with $\hat{u}$, $\hat{z}$ *the solutions of* (7) *and* (8) *respectively.*

Remark

- From above Theorem: $\hat{u}$, $\hat{z}$ are continuous, and the set $E := \{\hat{u}\hat{z} = 0\}$ is a closed subset of $\overline{\Omega}$.
- $\hat{m}\hat{u}\hat{z} \in BV(\Omega)$ gives that $\hat{m}$ belongs to $BV_{loc}(\Omega \setminus E)$.

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Example: $m \in [\alpha, \beta]$

$$\min \int_\Omega \left[j(x,u) + \psi(m) \right] dx,$$

subject to

$$\begin{cases} -\Delta u + mu = f & \text{in } \Omega \\ u \in H_0^1(\Omega). \end{cases}$$

We search $m \in [\alpha, \beta]$.

$$\psi(s) = \infty 1_{(-\infty,\alpha)\cup(\beta,\infty)}.$$

$$\mathrm{dom}(\psi) = [\alpha, \beta], \qquad \partial\psi(s) = \begin{cases} (-\infty, 0] & \text{if } s = \alpha \\ \{0\} & \text{if } \alpha < s < \beta \\ [0, \infty) & \text{if } s = \beta, \end{cases}$$

And,

$$h(\tau) = \begin{cases} \alpha & \text{if } \tau < 0 \\ \beta & \text{if } \tau \geq 0, \end{cases} \qquad \begin{cases} \hat{m} = \alpha & \text{a.e. in } \{\hat{u}\hat{z} < 0\} \\ \hat{m} = \beta & \text{a.e. in } \{\hat{u}\hat{z} > 0\} \\ \hat{m} \in [\alpha, \beta] & \text{a.e. in } \{\hat{u}\hat{z} = 0\}. \end{cases}$$

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Second COPI2A Meeting

Numerical Analysis

We propose the numerical resolution of problems

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- We consider $\Omega = B(0,1) \subset \mathbb{R}^2$.
- We apply a gradient descent method with projection.
- It depends on the function ψ associated to the volume constraint of the potential.

$$\psi(s) = \infty 1_{(-\infty,0)} + k|s|^2, \quad \text{first example},$$
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Algorithm

- Initialization: choose an admisible function $m_0 \in L^1(\Omega)$, such that $\Psi(m_0) < \infty$.
- For $n \geq 0$, iterate until stop condition as follows.
  - Compute u_n, z_n state and co-state associated to $\hat{m} = m_n$.
  - Compute $\tilde{m}_n$ descent direction associated to u_n and z_n, as:

$$\tilde{m}_n = \begin{cases} \dfrac{u_n z_n - 2km_n}{\|u_n z_n - 2km_n\|_{L^2(\Omega)}} & \text{in the first example,} \\ \operatorname{sgn}(u_n z_n) & \text{in the second example,} \end{cases}$$

  - Update the function m_n:

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First example

Taking $s_0 \in \mathbb{R}$ and

$$j(x,s) = \frac{1}{2}|s - s_0|^2, \qquad \psi(s) = \infty 1_{(-\infty,0)} + k|s|^2, \qquad f = 1.$$

For $m \equiv 0$, the solution of the state equation is

$$u(x) = \frac{1 - |x|^2}{4}.$$

Interesting case: $s_0 \in (0, 1/4)$

For $k = 0$ (ψ non-superlinear) not necessarily $\hat{m} \in L^1(\Omega)$:

$$\hat{m} = \frac{1}{s_0} 1_{\{|x|<a\}} dx + \frac{4s_0 - 1 + a^2(1 - 2\log a)}{4s_0 \log a} 1_{\{|x|=a\}} d\sigma,$$

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First example

$$\text{Optimal state } \hat{u}(x) = \begin{cases} s_0 & \text{if } |x| < a \\ \dfrac{1-|x|^2}{4} + \dfrac{(4s_0 - 1 + a^2)\log|x|}{4\log a} & \text{if } a \le |x| \le 1. \end{cases}$$

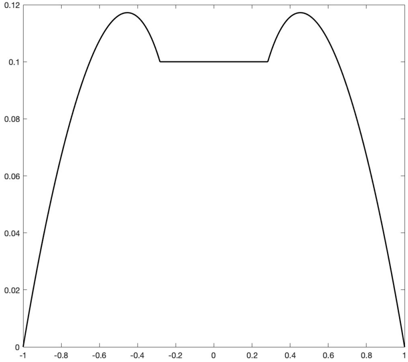


Figure: Example 1: Optimal state $\hat{u}$ for $k = 0$ (analytic solution) for $s_0 = 0.1$.

First example

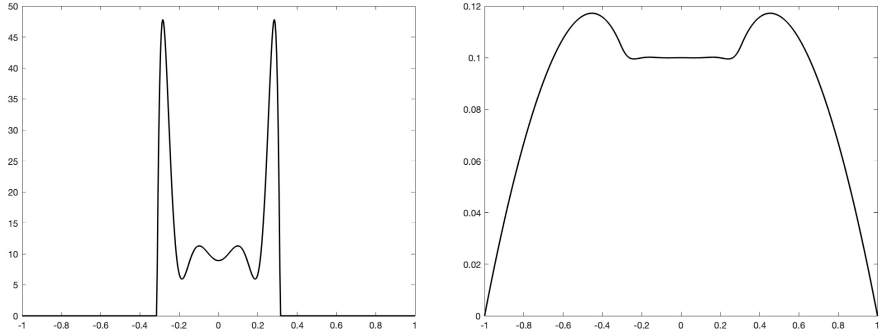
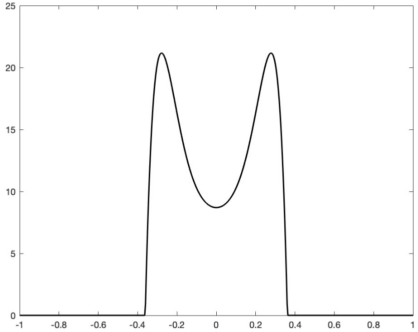


Figure: Optimal control $\hat{m}$ (left), and optimal state $\hat{u}$ (right), $k = 0$

First example



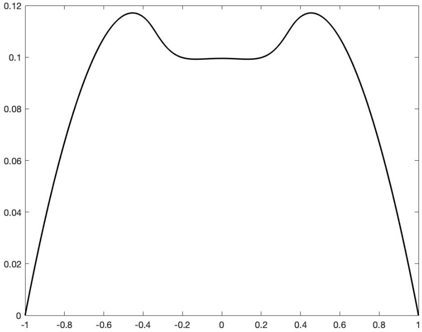


Figure: Optimal control $\hat{m}$ (left), and optimal state $\hat{u}$ (right), $k = 10^{-7}$

First example

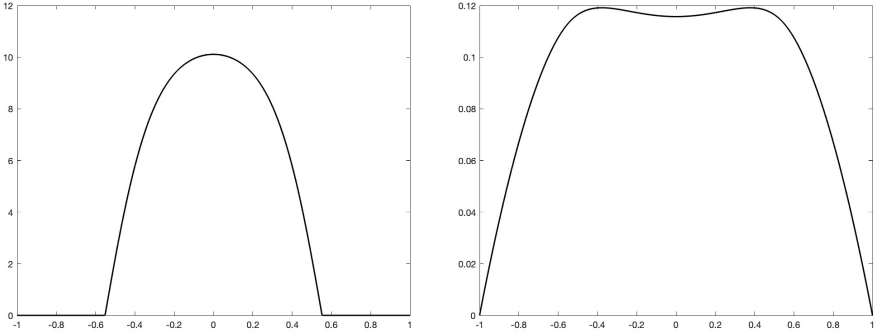


Figure: Optimal control $\hat{m}$ (left), and optimal state $\hat{u}$ (right), $k = 10^{-5}$

Second example

We consider

$$j(x,s) = g(x)s, \qquad \psi(s) = \infty 1_{(-\infty,\alpha)\cup(\beta,+\infty)},$$

with

$$g(x_1,x_2) = x_1^2 - x_2^2.$$

and

$$f(x_1,x_2) = 10(x_1^2 + x_2^2)\sin\left(13\arctan\left|\frac{x_2}{x_1}\right|\right) 1_{\{|x_1|>10^{-10}\}}.$$

We consider $\alpha = 0$, and we take three different values for β: 1, 10^2 and 10^4.

Second example

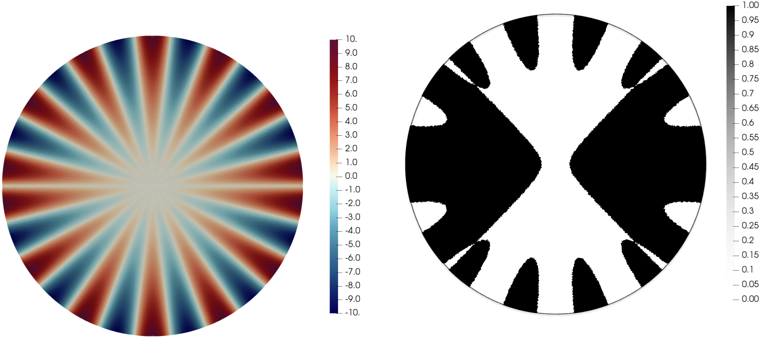


Figure: Right-hand side f (left). Optimal control $\hat{m}$ for $\beta = 1$ (right).

Second example

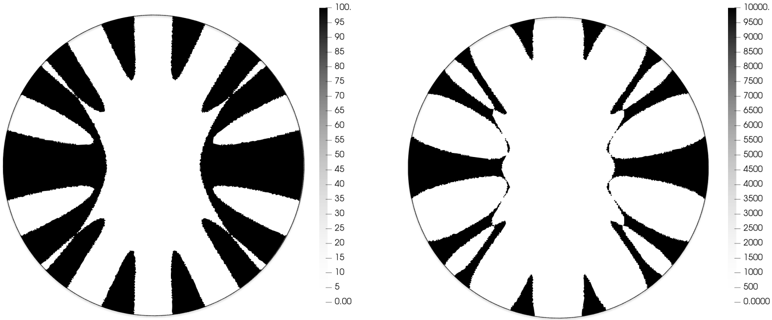


Figure: Optimal control $\hat{m}$ for $\beta = 10^2$ (left) and $\beta = 10^4$ (right).

THANK YOU

FOR YOUR

ATTENTION!!!