Nonlocal gradients and applications to Continuum Mechanics

José Carlos Bellido

Departamento de Matemáticas
Universidad de Castilla-La Mancha
Ciudad Real, SPAIN

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Let $u : \Omega \to \mathbb{R}^3$ be the deformation in the solid under the action of a distributed load F. Then for a given point in the **reference configuration** $x \in \Omega$, the function u maps the point to $u(x)$ in the **deformed configuration** $u(\Omega)$.

Hyperelasticity

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(this means that the Piola-Kirchhoff stress tensor is $T_R = \nabla W$) and therefore deformation under the action of a body force $F : \Omega \to \mathbb{R}^3$ must be minimizer of the functional

$$I(u) = \int_\Omega [W(\nabla u) - F \cdot u]\; dx.$$

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- *Infinite energy for infinite stretching:* $W(A) \to +\infty$ as $|A| \to +\infty$

Example of hyperelastic models

- **Mooney-Rivlin materials**:

$$W(A) = a|A|^2 + b|\operatorname{cof} A|^2 + J(\det A),$$

with $J : \mathbb{R} \to [0, +\infty]$ a function such that $J(t) = +\infty$ for any $t \leq 0$ and $\lim_{t\to+\infty} J(t) = +\infty$. a, $b > 0$.

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- **Ogden Materials**

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In a simplified situation in hyperelasticity we are therefore concerned with finding minimizers for an energy of the form

$$I(u) = \int_\Omega [h(\nabla u, \operatorname{cof} \nabla u, \det \nabla u) - F \cdot u] \, dx,$$

and $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$u = u_0 \text{ on } \partial\Omega.$$

Polyconvexity

Definition

$W : \mathbb{R}^{n\times n} \to \mathbb{R}$ is said polyconvex iff there exists a **convex** function $h : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R} \to \mathbb{R}$ such that

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Result

Polyconvexity implies **quasiconvexity**. Consequently, if $W(x, u, \cdot)$ polyconvex for a.e. $x \in \Omega$ and all $u \in \mathbb{R}^m$, then

$$I(u) = \int_\Omega W(x, u(x), \nabla u(x))\, dx$$

is **s.w.l.s.c.**

Polyconvexity

Weak continuity of the determinant

$n < p < \infty$. $u_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^n)$. Then

$$\det \nabla u_j \rightharpoonup \det \nabla u \text{ weakly } L^{\frac{p}{n}}(\Omega).$$

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Proof

- **Piola identity**: $\operatorname{div}(\operatorname{cof} \nabla u) = 0$
- $\det \nabla u$ may be written in divergence form:

$$\det \nabla u = \operatorname{div}(u_i(\operatorname{cof} \nabla u)^{(i)}) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(u_i(\, cof \nabla u)_{i,j})$$

- Integration by parts:

$$\int_\Omega (\det \nabla u_k)\varphi \, dx = - \int_\Omega (u_k)_i (\operatorname{cof} \nabla u_k)^{(i)} \nabla\varphi \, dx.$$

Theorem (John Ball 1977)

$n = 3$, $p \geq 2$, $q \geq p'$, $r > 1$, $c_1, c_2 > 0$. W such that:

- $W(A) = \hat{W}(F, \operatorname{cof} F, \det F)$ for all $F \in \mathbb{R}^{3\times 3}_+$, **polyconvex**.
- $W(A) \to \infty$ as $\det A \to 0$.
- $W(A) \geq c_1(|A|^p + |\operatorname{cof} A|^p + (\det A)^r) - c_2$.

Let

$$\mathcal{A}_{ad} = \left\{u \in W^{1,p} : \operatorname{cof} \nabla u \in L^q, \det \nabla u \in L^r, \det \nabla u > 0, u = u_0 \text{ on } \partial\Omega\right\}$$

and assume $\mathcal{A}_{ad} \neq \emptyset$. Then

$$I(u) = \int_\Omega [W(\nabla u) - F \cdot u] \, dx$$

admits minimizers

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Motivation

We explore **nonlocality** as a way to admit singular deformations in hyperelastic models set in spaces weaker than Sobolev

Nonlinear bond-based model

Peridynamics bond-base nonlinear energy:

$$E_{bb}(u) = \int_\Omega \int_{\Omega \cap B(x,\delta)} w(x, x - x', u(x) - u(x'))\, dx'\, dx - \int_\Omega F(x) \cdot u(x)\, dx.$$

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Question

Can model hyperelasticity with an energy like E_{bb} in such a way that the model converges to a local hyperelastic model as the *horizon* δ goes to zero?. And if the answer is positive, which local hyperleastic models can be obtained?

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We study this question in the framework of Γ-convergence, following the constructive process to obtain the (local) Γ-limit of the sequence E_{bb} as $\delta \to 0$[1]

[1] B., Mora-Corral, Pedregal, Hyperelasticity as a Γ- limit of Peridynamics when the horizon goes to zero. Calc. Var., 2015.

Nonlinear bond-based models

CONCLUSIONS AFTER LOCALIZATION OF BOND-BASED MODELS[2]:

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CONCLUSIONS AFTER LOCALIZATION OF BOND-BASED MODELS[2]:

- If we impose frame indifference and isotropy to the nonlinear energy E_{bb} and obtain the Γ-limit as $\delta \to 0$, very little local energies are recovered. Essentially variations fo the quadratic energy.

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- In particular, neither Mooney-Rivlin energies nor densities depending on the determinant or the cofactor can be recovered by the localization process.
- If we linearized a local model obtained from E_{bb} by localization, then Poisson ration is necessarily $\nu = \frac{1}{4}$.

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Nonlocal gradients

Given a function $u : \mathbb{R}^n \to \mathbb{R}$ and a kernel $\rho : \mathbb{R}^n \to \mathbb{R}_+$, the nonlocal ρ-gradient of u is given by the integral

$$D_\rho u(x) = \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|} \frac{x - y}{|x - y|} \rho(x - y) \, dy$$

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The idea is considering functional depending on nonlocal gradients rather that local gradients for modeling nonlinear elasticity:

$$I(u) = \int_\Omega W(x, u(x), D_\rho(x)) \, dx,$$

in the spirit of state based peridynamics

Riesz fractional gradient

Among nonlocal gradients, a **paradigmatic** examples is the Riesz s-fractional gradient:

$$D^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} \, dy$$

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$$H^{s,p}(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{H^{s,p}(\mathbb{R}^n)}}$$

with

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)} + \|D^s u\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)}.$$

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and the subspace of functions verifying an exterior Dirichlet condition

$$H_g^{s,p}(\Omega) = \{u \in H^{s,p} \ : \ u = g \text{ on } \Omega^c\}.$$

Riesz fractional gradient

For vector functions $u : \mathbb{R}^n \to \mathbb{R}^m$, its s-Riesz fractional gradient is defined as

$$D^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \otimes \frac{x - y}{|x - y|} \, dy,$$

and

$$H^{s,p}(\mathbb{R}^n, \mathbb{R}^m) = \left\{ u \in L^p(\mathbb{R}^n, \mathbb{R}^m) \ : \ D^s u \in L^p(\mathbb{R}^n, \mathbb{R}^{n \times m}) \right\},$$

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Riesz fractional gradient has an important drawback:

- it works only in the whole space $\mathbb{R}^n$

Fracture and cavitation

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- **Fracture:** Let $Q = (0,1)^n$ and $\varphi_2, \dots, \varphi_n \in C_c^\infty(\mathbb{R}^n)$. Define $u = (\chi_Q, \varphi_2, \dots, \varphi_n)$. Then

$$u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p < \frac{1}{s}, \quad \text{and} \quad u \notin H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p > \frac{1}{s}.$$

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- **Cavitation:** Let $\varphi \in C_c^\infty([0,\infty))$ be such that $\varphi(0) > 0$, and $u(x) = \frac{x}{|x|}\varphi(|x|)$. Then

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Fractional hyperelastic model

Given an stored-energy function $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$, such that $W(x, \lambda, \cdot)$ polyconvex for a.e. $x \in \Omega$ and all $\lambda \in \mathbb{R}^n$, we consider the following fractional hyperelastic energy

$$I(u) = \int_{\mathbb{R}^n} W(x, u(x), D^s u(x))\, dx,$$

with $u \in H^{s,p}_g(\Omega, \mathbb{R}^n)$.

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Since we are assuming polyconvexity, the main ingredient in order ingredient in order to apply the Direct Method of the CoV is the weak continuity of the determinant, or in general any minor, of $D^s u$ in $H^{s,p}$.

Calculus in $H^{s,p}$

Integration by parts:

Defining the fractional s-divergence of $\psi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ as

$$\operatorname{div}^s \psi(x) = -c_{n,s} \int_{\mathbb{R}^n} \frac{\psi(x) + \psi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} \, dy,$$

then if $u \in L^1_{loc}(\mathbb{R}^n)$ and $D^s u \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}^n)$ the integration by parts formula

$$\int_{\mathbb{R}^n} D^s u(x) \cdot \psi(x) \, dx = - \int_{\mathbb{R}^n} u(x) \operatorname{div}^s \psi(x) \, dx$$

holds.

Weak continuity of the minors

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Weak continuity of the minors

$u_j \rightharpoonup u$ weakly in $H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ and $M : \mathbb{R}^{n\times n} \to \mathbb{R}$ a minor or order r, then

$$M(D^s u_j) \rightharpoonup M(D^s u),$$

weakly in $L^{\frac{p}{s}}(\mathbb{R}^n)$.

Existence result

Theorem

$p \geq n-1$, $p > 1$ and $0 < s < 1$. Let
$W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ such that:

a) For a.e. $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^n$, the function $W(x, y, \cdot)$ is polyconvex.

b) Coercivity conditions on W (compatible with hyperelasticity)

$$\begin{cases} W(x,y,F) \geq a(x) + c\,|F|^p + c\,|\mathrm{cof}\, F|^q + h(|\det F|) \quad q > \frac{p^*}{p^*-1}, & \text{if } sp < \\ W(x,y,F) \geq a(x) + c\,|F|^p\,, & \text{if } sp \geq \end{cases}$$

for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ and all $F \in \mathbb{R}^{n\times n}$.

Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $u_0 \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$.
Then there exists a minimizer of

$$I(u) = \int_{\mathbb{R}^n} W(x, u(x), D^s u(x))\, dx$$

Γ-convergence

Convergence of fractional gradient

$u \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^m)$, then

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Compactness

$u_s \in H^{s,p}_g(\Omega)$, such that there exists $C > 0$ and $s_0 \in (0,1)$ with

$$\|D^s u_s\|_{L^p(\mathbb{R}^n,\mathbb{M}^{n\times m})} \leq C, \quad \forall s \in (s_0,1),$$

then there exists $u \in W^{1,p}(\mathbb{R}^n,\mathbb{R}^m)$ such that

$$D^s u_s \rightharpoonup \nabla u \text{ weakly in } L^p(\mathbb{R}^n,\mathbb{R}^m)$$

Γ-convergence

Γ-convergence result

$W : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$, such that $W(x, u, \cdot)$ is quasiconvex for a.e. $\mathbb{R}^n$ and all $u \in \mathbb{R}^m$. Let

$$I_s(u) = \int_{\mathbb{R}^n} W(x, u, D^s u)\, dx$$

be defined on $H_g^{s,p}(\Omega; \mathbb{R}^m)$, and let

$$I(u) = \int_{\mathbb{R}^n} W(x, u, \nabla u)\, dx$$

be defined on $W_g^{1,p}(\Omega; \mathbb{R}^m)$. Then

I_s Γ-converges to I.

Definition of the truncated Riesz fractional gradient

Inspired in the Riesz fractional gradient, given a bounded domain $\Omega \subset \mathbb{R}^n$, we define the following nonlocal gradient:

$$D^s_\delta u(x) = c_{n,s} \int_{B(x,\delta)} \frac{u(x) - u(y)}{|x - y|} \frac{x - y}{|x - y|} \frac{w(x - y)}{|x - y|^{n-1+s}} \, dy,$$

where w is a cut-off function.

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where w is a cut-off function.

Now, we define de space $H^{s,p,\delta}(\Omega)$ as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_{H^{s,p,\delta}(\Omega)} = \left(\|u\|_{L^p(\Omega_\delta)}^p + \|D_\delta^s u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}},$$

where Ω_δ is the union of Ω with a tubular neighbourhood of the boundary of radius δ.

Nonlocal gradient in bounded domains

Observed that

$$\|D_\delta^s u\|_{L^p(\Omega)} = c_{n,s} \left(\int_\Omega \left| \int_{B(x,\delta)} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{w(x-y)}{|x-y|^{n-1+s}} \, dy \right|^p \, dx \right)$$

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which is quite different of typical double-integral seminorms, like Gagliardo seminorm f.i. Therefore, new ideas are required in order to obtain structural properties of the space $H^{s,p,\delta}(\Omega)$.

Nonlocal gradients in bounded domains

Integration by parts

Defining the nonlocal divergence as

$$\operatorname{div}_\delta^s \phi(x) = -\operatorname{pv}_x c_{n,s} \int_{B(x,\delta)} \frac{\phi(x)+\phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n-1+s}} \, dy.$$

Suppose that $u \in C_c^\infty(\mathbb{R}^n)$ and $\phi \in C_c^1(\Omega, \mathbb{R}^n)$. Then $D_\delta^s u \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and $\operatorname{div}_\delta^s \phi \in L^\infty(\mathbb{R}^n)$. Moreover,

$$\int_\Omega D_\delta^s u(x) \cdot \phi(x) \, dx = -\int_\Omega u(x) \operatorname{div}_\delta^s \phi(x) \, dx$$

$$-(n-1+s) \int_\Omega \int_{\Omega_\delta \setminus \Omega} \frac{u(y)\phi(x)}{|x-y|} \cdot \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n-1+s}} \, dy \, dx,$$

and these three integrals are absolutely convergent.

New fractional space in bounded domains

The space $H^{s,p,\delta}(\Omega)$ must fulfill two structural features in order to be useful for variational analysis, in particular for establishing new models of nonlocal hyperelasticity:

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The space $H^{s,p,\delta}(\Omega)$ must fulfill two structural features in order to be useful for variational analysis, in particular for establishing new models of nonlocal hyperelasticity:

- A Poincaré inequality
- Compact embedding into L^p

The key result

Nonlocal fundamental theorem of calculus

There exits a function $V_\delta^s \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ such that for every $u \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$u(x) = \int_{\mathbb{R}^n} D_\delta^s u(y) \cdot V_\delta^s(x-y)\, dy.$$

Moreover, for every $R > 0$ there exists $M > 0$ such that

$$|V_\delta^s(x)| \leq \frac{M}{|x|^{n-s}}, \qquad x \in B(0,R) \setminus \{0\}.$$

Main points in the proof

1. $\varphi(x) = \nabla\varphi * \left(\sigma_{n-1}^{-1} \frac{\cdot}{|\cdot|}\right);$

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   for all $u \in C_c^\infty(\mathbb{R}^n)$;
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4. Show V_δ^s is indeed a function, and find estimates on it;
5. Apply the first item to obtain the representation theorem.

Consequences of NFTC

Poincaré inequality

Let $1 < p < \infty$. Then there exists $C > 0$ such that for all $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$,

$$\|u\|_{L^p(\Omega)} \leq C \left\| D_\delta^s u \right\|_{L^p(\Omega)} .$$

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We can also show Sobolev, Morrey, Trudinger and Hardy type inequalities in $H^{s,p,\delta}(\Omega)$.

Consequences of NFTC

Estimation on the translations

a) Let $1 < p < \infty$. Then there exists $C > 0$ such that for all $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$ and $h \in \mathbb{R}^n$,

$$\left(\int_\Omega |u(x+h) - u(x)|^p \, dx \right)^{\frac{1}{p}} \leq C \, |h|^s \, \| D_\delta^s u \|_{L^p(\Omega)} . \tag{1}$$

b) Let $p = 1$. Then for all $M > 0$ there exists $C > 0$ such that for all $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$ and $h \in B(0, M)$, inequality (1) holds.

Consequences of NFTC

Compact embeddings

Let $g \in H^{s,p,\delta}(\Omega)$. Then, for any sequence $\{u_j\}_{j\in\mathbb{N}} \subset H^{s,p,\delta}_g(\Omega_{-\delta})$ such that

$$u_j \rightharpoonup u \quad \text{in } H^{s,p,\delta}(\Omega),$$

for some $u \in H^{s,p,\delta}(\Omega)$, one has $u \in H^{s,p,\delta}_g(\Omega_{-\delta})$ and:

1. if $p > 1$, $u_j \to u \quad$ in $L^q(\Omega)$, for every q satisfying

$$\begin{cases} q \in [1, p^*_s) & \text{if } sp < n, \\ q \in [1, \infty) & \text{if } sp = n, \\ q \in [1, \infty] & \text{if } sp > n. \end{cases}$$

2. if $p = 1$,

$$u_j \to u \quad \text{in } L^1(\Omega).$$

Nonlocal hyperelasticity

Theorem

Let $p \geq n-1$, $p > 1$, $q \geq \frac{n}{n-1}$. Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ such that:

a) For a.e. $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^n$, the function $W(x, y, \cdot)$ is polyconvex.

b) Coercivity conditions on W (compatible with hyperelasticity)

Let $u_0 \in H^{s,p,\delta}(\Omega, \mathbb{R}^n)$.Then there exists a minimizer of

$$I(u) = \int_\Omega W(x, u(x), D^s_\delta u(x))\, dx$$

Then there exists a minimizer of I in $H^{s,p,\delta}_{u_0}(\Omega_{-\delta}, \mathbb{R}^n)$.

Fractional and nonlocal linear elasticity

By linearization of the previous fractional and nonlocal (*polyconvex*) hyperelasticity models we obtain respectively the following fractional and nonlocal linear elasticity models:

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We show existence of solutions for these problems via a nonlocal Korn's inequality.

Connection with Eringen's nonlocal elasticity model

Erigen's model of nonlocal elasticity

$$\begin{cases} -\operatorname{div}\sigma = f, & \text{in } \Omega \\ \sigma(x) = \displaystyle\int_\Omega A(x,x')\,C D_{\text{sym}} v(x')\,dx', & \text{in } \Omega \\ v = \bar{v}, & \text{on } \partial\Omega. \end{cases}$$

with $A(\cdot,\cdot)$ a given nonlocal convolution kernel.

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2. **Nonlocal case:** $A(x,x') = Q_\delta^s * Q_\delta^s(|x-x'|)$.

Work in progress

- Nonlocal vector calculus for D_δ^s (*with P. Radu, M. Foss and J. Cueto*):

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- Nonlocal vector calculus for D_δ^s (*with P. Radu, M. Foss and J. Cueto*):
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- Generalization of the study of of nonlocal gradients and associated Bessel-type spaces to general kernels (*with C. Mora-Corral and H. Schönberger*):

$$D_\rho u(x) = \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho(x-y)\, dy.$$

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  2. Equivalence of spaces for different ρ's

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