Bilinear control of evolution equations. Part 1

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Control systems

In a given Banach space X

Dynamical system: $$u' = f(u, \mathbf{p})$$

↑ **control function**

where

- $u : [0, T] \to X$ is the state variable
- $\mathbf{p}$ is the control

Additive control for linear systems

$$\begin{cases} u'(t) + Au(t) + B\mathbf{p}(t) = 0 & t \in [0, T] \\ u(0) = u_0 \end{cases}$$

where

- $A : D(A) \subset X \to X$ with $-A$ the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on X
- e^{-tA} $(t \geqslant 0)$ is the semigroup generated by $-A$
- $B : D(B) \subset X \to X$ is a linear operator on X that can be either bounded or unbounded
- $\mathbf{p} : [0, T] \to X$ is the control

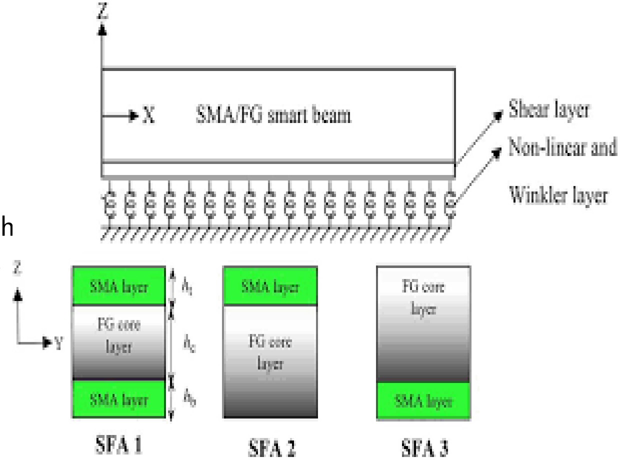
Scalar-input bilinear control systems

Motivations

Bilinear controls enter the system equations as coefficients changing (at least some of) the principal parameters of the process at hand

Examples

- by embedded *smart* alloys, the natural frequency response of a beam can be changed
- the rate of a chemical reaction can be altered by various catalysts and/or by th speed at which the reaction ingredients are mechanically mixed

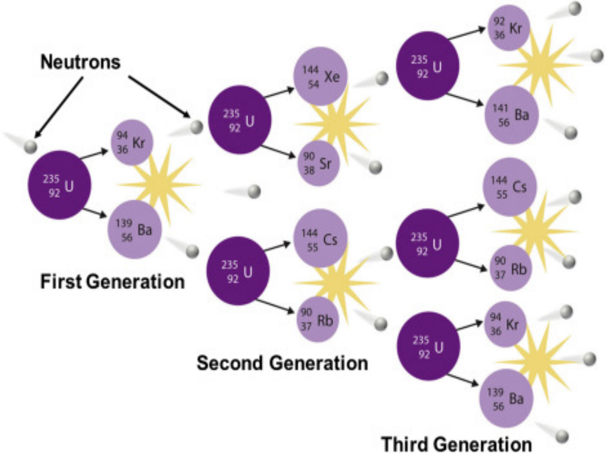


A simplified model of a nuclear chain reaction

A chain reaction refers to a process in which neutrons released in fission produce an additional fission in at least one further nucleus. This nucleus in turn produces neutrons, and the process repeats. The process may be controlled (nuclear power) or uncontrolled (nuclear weapons).

$$\boxed{u_t = a^2 \Delta u + v(t,x) u}$$

- $u(t,x) \geq 0$ neutron density in the reaction
- $v(t,x) > 0$ neutron amount in the surrounding medium
- $v(t,x)u$ neutrons provided by the collision of the particles in the reaction with the surrounding medium

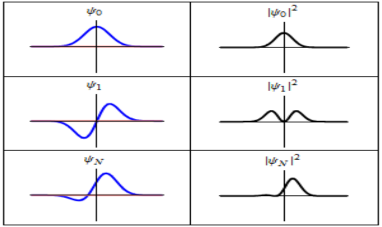


Schrödinger equation

The Schrödinger equation is a linear partial differential equation that describes the wave function or state function of a quantum-mechanical system

$$\boxed{i\psi_t = -\Delta\psi - p(t)\mu(x)\psi}$$

- ψ wave function of a particle
- p amplitude of the electric field
- μ dipolar moment of the particle



Fokker-Planck equation

Let X_t be a $1D$ duffusion process in $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$

$$dX_t = b(t, X_t)dt + \sigma dW_t$$

where W_t is the standard Wiener process

The probability density $u(t, \cdot)$ of X_t

$$\mathbb{P}(\alpha \leq X_t \leq \beta) = \int_\alpha^\beta u(t,x)dx$$

satisfies the Fokker-Planck equation

$$u_t(t,x) - \frac{\sigma^2}{2} u_{xx}(t,x) + \big(b(t,x)u(t,x)\big)_x = 0 \qquad (FP)$$

Problem: to steer the initial density u_0 of X_0 to a given target density u_T by using a drift $b(t,x)$ for (FP) of the form $b(t,x) = p(t)\mu(x)$

The abstract model

Systems where control enters as a coefficient

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0 & t \in [0, T] \\ u(0) = u_0 \in X \end{cases}$$

- the state space $(X, \langle \cdot, \cdot \rangle)$ is a separable Hilbert space
- $A : D(A) \subset X \to X$ with $-A$ the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on X
- e^{-tA} $(t \geqslant 0)$ is the semigroup generated by $-A$
- $B : D(B) \subset X \to X$ is a linear operator on X that can be either bounded or unbounded
- control $p \in L^2(0, T)$ is a square summable scalar function

What are the difficulties?

The map $\boldsymbol{\Phi} : \mathbf{p} \mapsto u$ is

Additive control:

$$\begin{cases} u' + Au + B\mathbf{p} = 0 \\ u(0) = u_0 \end{cases}$$

Bilinear control:

$$\begin{cases} u' + Au + \mathbf{p}Bu = 0 \\ u(0) = u_0 \end{cases}$$

What are the difficulties?

The map $\mathbf{\Phi} : \mathbf{p} \mapsto u$ is

Additive control:

$$\begin{cases} u' + Au + B\mathbf{p} = 0 \\ u(0) = u_0 \end{cases}$$

$\downarrow$

linear

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-\tau)A}Bp(\tau)d\tau$$

$$\|u(t)\| \leqslant C_T\big(\|u_0\| + \|p\|_{L^1(0,T;X)}\big)$$

Bilinear control:

$$\begin{cases} u' + Au + \mathbf{p}Bu = 0 \\ u(0) = u_0 \end{cases}$$

$\downarrow$

nonlinear

$$u(t) = e^{-tA}u_0 - \int_0^t p(\tau)e^{-(t-\tau)A}Bu(\tau)ds$$

$$\|u(t)\| \leqslant C_T\|u_0\|e^{C_T\|p\|_{L^1(0,T;X)}}$$

An obstruction to exact controllability

Bilinear control:

$$\begin{cases} u' + Au + \mathbf{p}Bu = 0 \\ u(0) = u_0 \end{cases} \tag{1}$$

Let $u_0 \in X$ and denote by $u(\cdot; p, u_0)$ the unique solution of (1) for $p \in L^1_{loc}(0, \infty)$. Define the attainable set from u_0 by

$$S(u_0) := \left\{ u(t; p, u_0); t \geq 0, p \in L^1_{loc}(0, \infty) \right\}$$

Theorem (Ball, Marsden, Slemrod 1982)

Let $B \in \mathcal{L}(X)$. If $\dim X = \infty$, then $X \setminus S(u_0)$ is dense

Consequently, $S(u_0) \subsetneq X$ and (1) fails to be exactly controllable

Multiplicative control for parabolic systems

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Scalar single-input bilinear control for the Schrödinger and wave equations

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  - F. Alabau-Boussouira, P. Cannarsa, and C. Urbani. Exact controllability to eigensolutions for evolution equations of parabolic type via bilinear control, *Nonlinear Differ. Equ. Appl.* (2022), arXiv:2105.05732
  - F. Alabau-Boussouira, P. Cannarsa, and C. Urbani. Bilinear control of evolution equations with unbounded lower order terms. Application to the Fokker-Planck equation, arXiv:2303.04465

Bilinear control and preservation of energy

Given a domain $\mathcal{O} \subset \mathbb{R}^n$ and $u_0 \in H^1_0(\mathcal{O})$, find $p : [0, \infty) \to \mathbb{R}$ such that the solution to

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + p(t)u(t,x) & \text{in } \mathbb{R}_+ \times \mathcal{O} \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\mathcal{O} \\ u(0,x) = u_0(x) & \xi \in \mathcal{O} \end{cases} \tag{CL}$$

satisfies $\|u(t)\|_{L^2(\mathcal{O})} = \|u_0\|_{L^2(\mathcal{O})} \qquad \forall t \geqslant 0$

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Controllability to eigensolutions

Assumptions

Let $(X, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space and $\boldsymbol{A} : D(\boldsymbol{A}) \subset X \to X$ a densely defined linear operator satisfying the following Standing Assumptions

$$\begin{array}{ll} (a) & \boldsymbol{A} \text{ is self-adjoint} \\ (b) & \exists \sigma \geq 0 \;:\; \langle \boldsymbol{A}x, x \rangle \geq -\sigma \|x\|^2 \;,\; \forall x \in D(\boldsymbol{A}) \\ (c) & D(\boldsymbol{A}) \subseteq X \text{ is compact} \end{array} \qquad (SA)$$

Assumptions

Let $(X, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space and $A : D(A) \subset X \to X$ a densely defined linear operator satisfying the following Standing Assumptions

$$\begin{array}{ll} (a) & A \text{ is self-adjoint} \\ (b) & \exists \sigma \geq 0 \ : \ \langle Ax, x \rangle \geq -\sigma \|x\|^2 \ , \ \forall x \in D(A) \\ (c) & D(A) \subseteq X \text{ is compact} \end{array} \qquad (SA)$$

$$\downarrow$$

1. X has a complete orthonormal system $\{\varphi_k\}_{k \in \mathbb{N}^*}$ of eigenvectors of A
2. the eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}^*}$ of A satisfy $-\sigma \leq \lambda_k \to +\infty$ as $k \to +\infty$
3. $-A$ generates the strongly continuous semigroup e^{-tA}

The state equation

Given $T > 0$, consider the bilinear control problem

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, \quad t \in [0, T] \\ u(0) = u_0 \end{cases} \qquad (\star)$$

where $B \in \mathcal{L}(X)$ and $p \in L^2(0, T)$

The state equation

Given $T > 0$, consider the bilinear control problem

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0, T] \\ u(0) = u_0 \end{cases} \qquad (\star)$$

where $B \in \mathcal{L}(X)$ and $p \in L^2(0, T)$

Consider system $(\star)$ with $p = 0$ and $u_0 = \varphi_j$ $(j \geq 1)$

$$\begin{cases} u'(t) + Au(t) = 0, & t \in [0, T] \\ u(0) = \varphi_j \end{cases}$$

Any solution $\boxed{\psi_j(t) = e^{-\lambda_j t}\varphi_j}$ is called an **eigensolution**

For $j = 1$, the solution $\boxed{\psi_1(t) = e^{-\lambda_1 t}\varphi_1}$ is the **ground state solution**

j-null controllable pairs

Definition

Let $T > 0$ and $j \geq 1$. The pair $\{A, B\}$ is called j-null controllable in time T if there exists a constant $N_T > 0$ such that for every $y_0 \in X$ one can find a control $p \in L^2(0, T)$ satisfying

$$||p||_{L^2(0,T)} \leq N_T ||y_0||,$$

for which $y(T; y_0, p) = 0$, where $y(\cdot; y_0, p)$ is the solution of

$$\begin{cases} y'(t) + Ay(t) + p(t)B\varphi_j = 0, & t \in [0, T] \\ y(0) = y_0 \end{cases}$$

The pair $\{A, B\}$ is called j-null controllable if there exists $T_0 > 0$ such that $\{A, B\}$ is j-null controllable in time T_0

The *control cost* is given by $N_j(T) = \sup_{\|y_0\|=1} \inf_{y(T;y_0,p)=0} \|p\|_{L^2(0,T)}$

Local exact controllability to eigensolutions

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0 & (t > 0) \\ u(0) = u_0 \end{cases} \qquad (\star)$$

Theorem

Assume that $\{A, B\}$ is j-null controllable $(j \geq 1)$ in any time $T > 0$ and $\exists \nu, T_0 > 0$ such that

$$N_j(\tau) \leq e^{\nu/\tau} \quad \forall\, 0 < \tau \leq T_0$$

Then for any $T > 0$ there exists $R_T > 0$ such that for any $u_0 \in B_{R_T}(\varphi_j)$ there exists a control $p \in L^2(0, T)$ for which $\boldsymbol{u(T) = e^{-\lambda_j T}\varphi_j}$*. Moreover,*

$$R_T = R_T\big(\nu, \sigma, \|B\|, T, T_0\big) \quad \text{and} \quad \|p\|_2 \leqslant C\big(\nu, \sigma, \|B\|, T, T_0\big)$$

A Newton type algorithm ($j = 1, \lambda_1 = 0$)

Step 1: linearization

For $T > 0$ consider the systems

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0, T] \\ u(0) = u_0 \end{cases} \qquad \begin{cases} \psi_1'(t) + A\psi_1(t) = 0, & t \in [0, T] \\ \psi_1(0) = \varphi_1 \end{cases}$$

Note that $\psi_1(t) \equiv \varphi_1$ and set $\boxed{v := u - \varphi_1}$ Then linearize to obtain

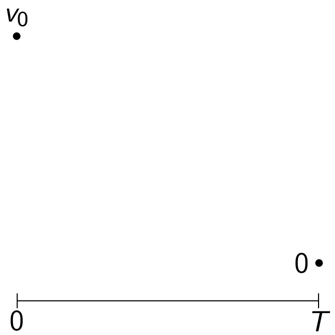
$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0 \\ v(0) = v_0 = u_0 - \varphi_1 \end{cases} \qquad \begin{cases} y(t)' + Ay(t) + p(t)B\varphi_1 = 0 \\ y(0) = v_0 \end{cases}$$

A Newton type algorithm ($j = 1, \lambda_1 = 0$)

Step 2: induction procedure $[0, T]$

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0 \\ v(0) = v_0 = u_0 - \varphi_1 \end{cases}$$

$$\begin{cases} y(t)' + Ay(t) + p(t)B\varphi_1 = 0 \\ y(0) = v_0 \end{cases}$$



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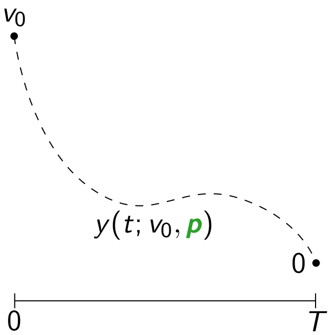

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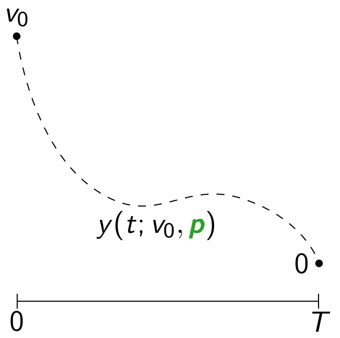
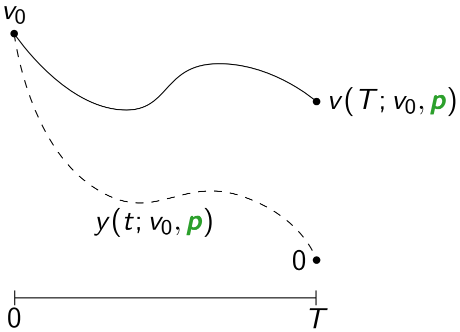


$$||p||_{L^2(0,T)} \leq N_T ||v_0||$$

A Newton type algorithm ($j = 1, \lambda_1 = 0$)

Step 2: induction procedure $[0, T]$

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \\ v(0) = v_0 = u_0 - \varphi_1, \end{cases} \qquad \begin{cases} y(t)' + Ay(t) + p(t)B\varphi_1 = 0, \\ y(0) = v_0. \end{cases}$$



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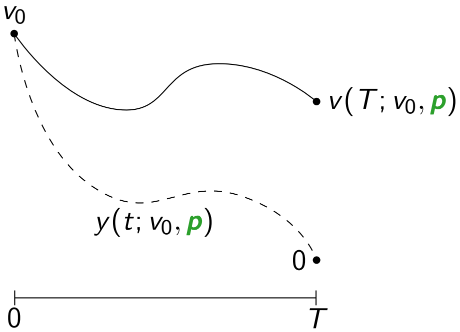


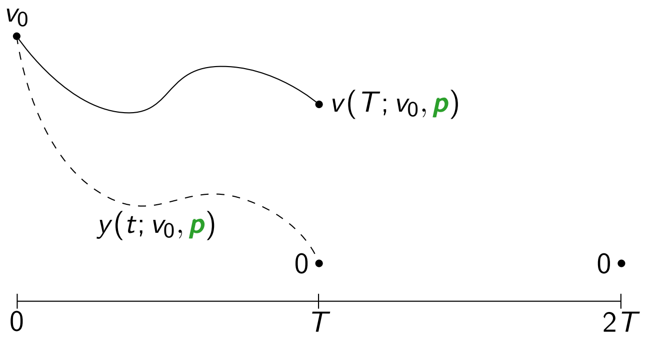
$$||p||_{L^2(0,T)} \leq N_T||v_0|| \qquad ||(v-y)(T)|| = ||v(T)|| \leq K_T||v_0||^2 = \frac{1}{K_T}(K_T||v_0||)^2$$

A Newton type algorithm ($j = 1, \lambda_1 = 0$)

Step 2: induction procedure $[T, 2T]$

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0 \\ v(T) = v_T \quad \text{(given by step 1)} \end{cases}$$

$$\begin{cases} y(t)' + Ay(t) + p(t)By(t) + p(t)B\varphi_1 = 0 \\ y(T) = v_T \end{cases}$$



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Step 2: induction procedure $[T, 2T]$

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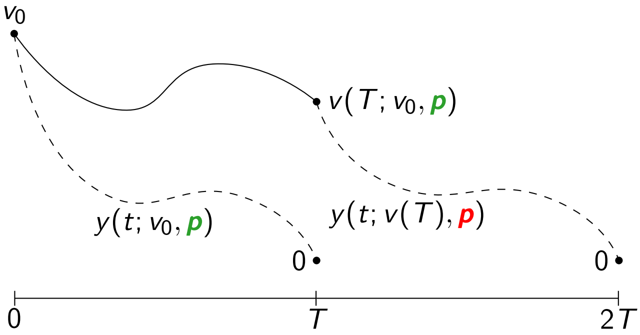

A Newton type algorithm ($j = 1, \lambda_1 = 0$)

Step 2: induction procedure $[T, 2T]$

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0 \\ v(T) = v_T \end{cases}$$

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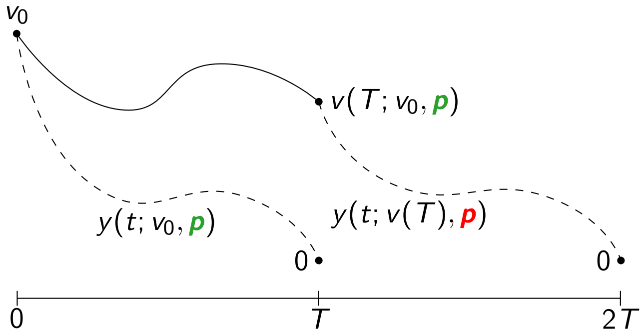
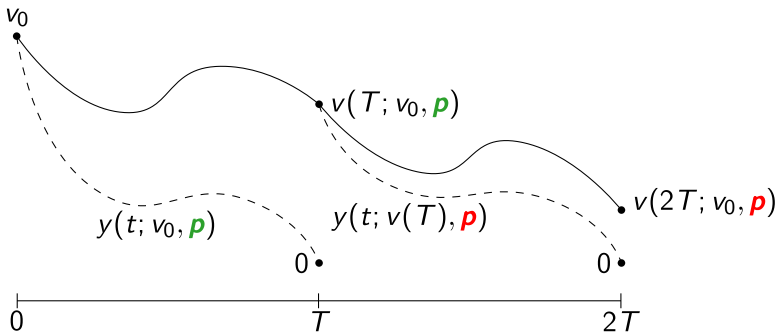


$$||p||_{L^2(T,2T)} \leq N_T ||v(T)||$$

A Newton type algorithm ($j = 1, \lambda_1 = 0$)

Step 2: induction procedure $[T, 2T]$

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \\ v(T) = v_T, \end{cases} \qquad \begin{cases} y(t)' + Ay(t) + p(t)B\varphi_1 = 0, \\ y(T) = v_T. \end{cases}$$



$$||p||_{L^2(T,2T)} \leq N_T||v(T)||$$

A Newton type algorithm ($j = 1, \lambda_1 = 0$)

Step 2: induction procedure $[T, 2T]$

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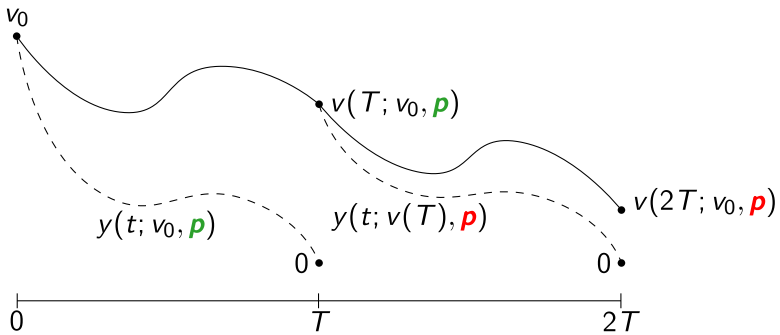
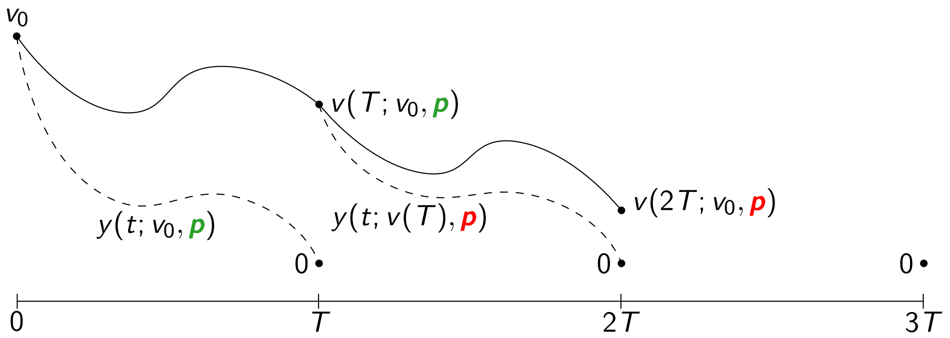


$$\|p\|_{L^2(T,2T)} \le N_T \|v(T)\|, \quad \|(v-y)(2T)\| = \|v(2T)\| \le K_T \|v(T)\|^2 \le \frac{1}{K_T}(K_T\|v_0\|)^{2^2}$$

A Newton type algorithm ($j = 1, \lambda_1 = 0$)

Step 2: induction procedure $[2T, 3T]$

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \\ v(2T) = v_{2T}, \end{cases} \qquad \begin{cases} y(t)' + Ay(t) + p(t)B\varphi_1 = 0, \\ y(2T) = v_{2T}. \end{cases}$$



A Newton type algorithm ($j = 1, \lambda_1 = 0$)

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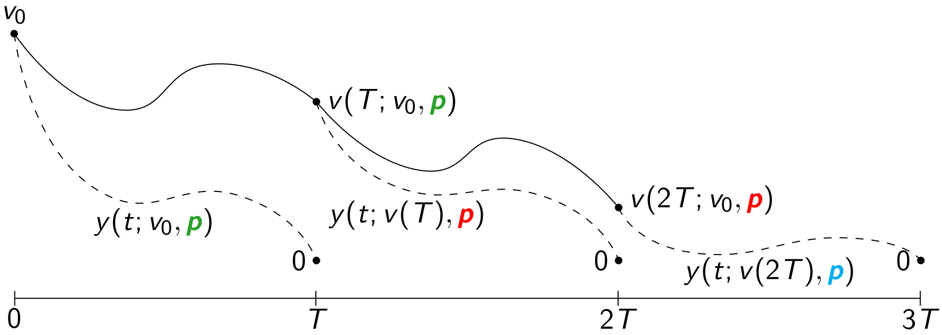

A Newton type algorithm ($j = 1, \lambda_1 = 0$)

Step 2: induction procedure $[2T, 3T]$

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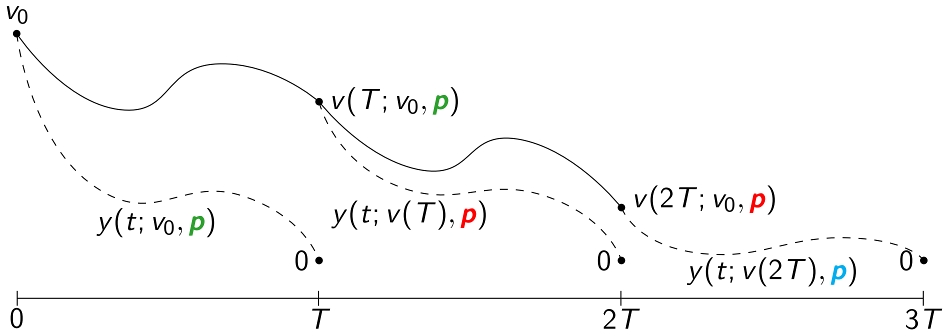

$$||p||_{L^2(2T,3T)} \leq N_T ||v(2T)||$$

A Newton type algorithm ($j = 1, \lambda_1 = 0$)

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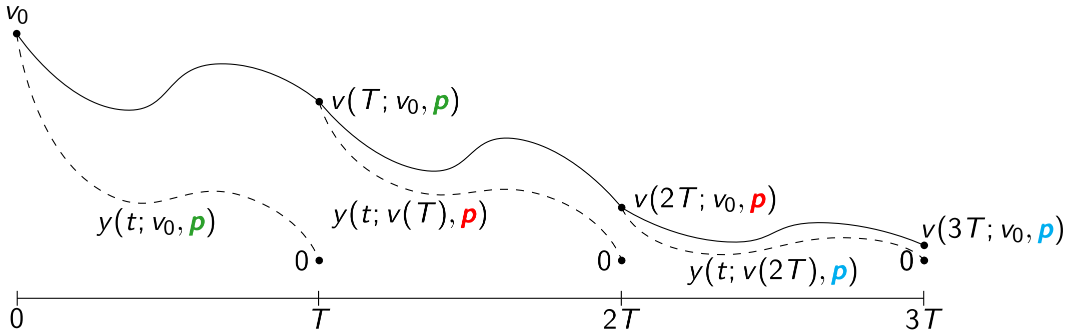

$$||p||_{L^2(2T,3T)} \leq \Lambda_T ||v(2T)||$$

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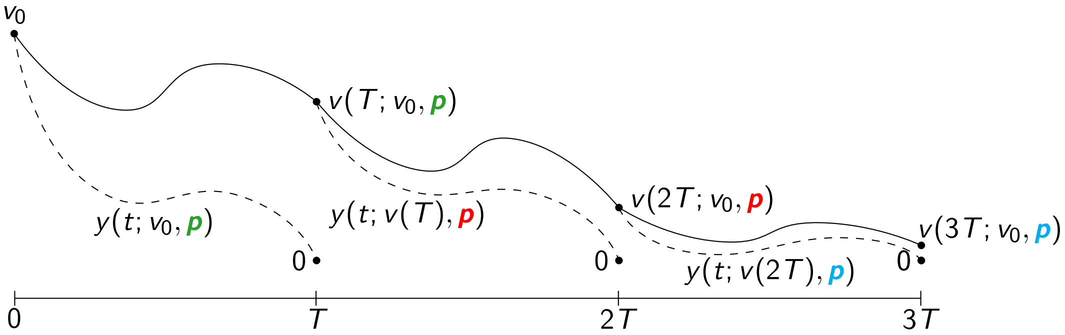

$$\|p\|_{L^2(2T,3T)} \leq \Lambda_T \|v(2T)\| \qquad \|(v-y)(3T)\| = \|v(3T)\| \leq K_T \|v(2T)\|^2 \leq \frac{1}{K_T}(K_T\|v_0\|)^{2^3}$$

The right time scaling

For fixed $T > 0$, we construct a solution v of

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \quad t \in [0, \frac{\pi^2}{6}T] \\ v(0) = v_0 = u_0 - \varphi_1 \end{cases} \tag{6}$$

by applying the Newton type algorithm on consecutive intervals $[\tau_n, \tau_{n+1}]$ with

$$\tau_n = \sum_{j=1}^{n} \frac{T}{j^2}$$

Technical estimates ensure that

$$||v(\tau_n)|| \leq \left(e^{C/T}||v_0||\right)^{2^n} \longrightarrow 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad e^{C/T}||v_0|| < 1$$

and this yields the conclusion $v(\frac{\pi^2}{6}T) = 0$

Conditions for j-null controllability

Sufficient conditions for j-null controllability

Theorem

In addition to (SA) suppose that

1. $A : D(A) \subset X \to X$ *satisfies the gap condition*

$$\sqrt{\lambda_{k+1} - \lambda_1} - \sqrt{\lambda_k - \lambda_1} \geq \alpha, \quad \forall\, k \in \mathbb{N}^* \tag{G}$$

   for some constant $\alpha > 0$

2. $B : X \to X$ *is a bounded linear operator such that*

$$(i)\ \langle B\varphi_j, \varphi_j \rangle \neq 0 \qquad (ii)\ \exists \beta, q > 0\ :\ |\lambda_k - \lambda_j|^q\, |\langle B\varphi_j, \varphi_k \rangle| \geq \beta, \quad \forall\, k \neq j \tag{S_j}$$

Then, the pair $\{A, B\}$ *is j-null controllable in any time* $T > 0$ *and* $\exists \nu, T_0 > 0$ *such that*

$$N_j(\tau) \leq e^{\nu/\tau} \quad \forall\, 0 < \tau \leq T_0$$

Comments on sufficient conditions

- The gap condition (G) is due to the fact that we restrict the analysis to single-input controls $p(t)$. It is satisfied by several control systems in one space dimension

Comments on sufficient conditions

- The gap condition (G) is due to the fact that we restrict the analysis to single-input controls $p(t)$. It is satisfied by several control systems in one space dimension
- The spreading condition (S_j)

$$\langle B\varphi_j, \varphi_k\rangle \neq 0, \quad \forall\, k \in \mathbb{N}^*$$

is necessary for the null controllability of the linear system

$$\begin{cases} y'(t) + Ay(t) + p(t)B\varphi_j = 0, & t \in [0, T] \\ y(0) = y_0 \end{cases}$$

Necessity of (S_j)

Proof

The identity $y(T) = 0$ is equivalent to

$$\sum_{k\in\mathbb{N}^*} \langle y_0, \varphi_k\rangle e^{-\lambda_k T}\varphi_k = \int_0^T p(s) \sum_{k\in\mathbb{N}^*} \langle B\varphi_j, \varphi_k\rangle e^{-\lambda_k(T-s)}\varphi_k ds.$$

Since $\{\varphi_k\}_{k\in\mathbb{N}^*}$ is an othonormal basis of X, we have that

$$\langle y_0, \varphi_k\rangle = \int_0^T e^{\lambda_k s} p(s)\langle B\varphi_j, \varphi_k\rangle ds, \quad \forall\, k \in \mathbb{N}^*.$$

In particular, if (S_j) is violated, there exists some $\bar{k} \in \mathbb{N}^*$ such that $\langle B\varphi_j, \varphi_{\bar{k}}\rangle = 0$. Hence, in the $\bar{k}$-th direction we have that

$$\langle y_0, \varphi_{\bar{k}}\rangle = \int_0^T e^{\lambda_{\bar{k}} s} p(s)\langle B\varphi_j, \varphi_{\bar{k}}\rangle ds = 0$$

which yields $y_0 \in \varphi_{\bar{k}}^{\perp}$.

Proof of sufficient conditions ($j = 1$)

Let $T > 0$ and $y_0 \in X$. We want to find a control $p \in L^2(0,T)$ such that

$$y(T) = 0 \quad \text{and} \quad ||p||_{L^2(0,T)} \leq N_T ||y_0||$$

where

$$\begin{cases} y(t)' + Ay(t) + p(t)B\varphi_1 = 0 \\ y(0) = y_0 \end{cases}$$

Since $y(t) = e^{-tA}y_0 - \int_0^t e^{-(t-s)A}p(s)B\varphi_1 ds$,

$$y(T) = 0 \iff \sum_{k\geq 1} \langle y_0, \varphi_k \rangle e^{-\lambda_k T}\varphi_k = \int_0^T p(s) \sum_{k\geq 1} \langle B\varphi_1, \varphi_k \rangle e^{-\lambda_k (T-s)}\varphi_k ds.$$

or

$$\int_0^T e^{\lambda_k s} p(s) ds = \frac{\langle y_0, \varphi_k \rangle}{\langle B\varphi_1, \varphi_k \rangle} \quad (k \geq 1)$$

Proof of sufficient conditions ($j = 1$)

The moment method

Thanks to (G) there exists a biorthogonal family[1] $\{\sigma_j\}_{j\geq 1}$ to $\{e^{\lambda_k t}\}_{k\geq 1}$ in $L^2(0,T)$, that is,

$$\forall k, j \geq 1, \quad \int_0^T \sigma_j(t) e^{\lambda_k t} dt = \delta_{jk}$$

Moreover

$$||\sigma_j||^2_{L^2(0,T)} \leq C^2_\alpha(T) e^{-2\lambda_j T} e^{C_\alpha \sqrt{\lambda_j}}, \quad \forall j \geq 1 \qquad (\star\star)$$

Therefore

$$p(s) = \sum_{k=1}^{\infty} \frac{\langle v_0, \varphi_k \rangle}{\langle B\varphi_1, \varphi_k \rangle} \sigma_k(s) \quad \Longrightarrow \quad \int_0^T e^{\lambda_k s} p(s) ds = \frac{\langle v_0, \varphi_k \rangle}{\langle B\varphi_1, \varphi_k \rangle}$$

$(\star\star)$ and (S_1) ensure that the above series converges in $L^2(0,T)$ and

$$||p||_{L^2(0,T)} \leq e^{C/T} ||v_0||$$

[1] See [P. Cannarsa, P. Martinez, J. Vacostenoble, Math. Control Relat. Fields (2017)], see also Fattorini and Russell [1971], Tennebaum and Tucsnak [2007], and Lissy?[2014, 2015]

Applications

Heat equation with Dirichlet boundary conditions

Consider the bilinear control system

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)\mu(x)u(t,x) = 0, & (t,x) \in (0,\infty)\times(0,1) \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

which can be recast as the abstract evolution equation in $X = L^2(0,1)$

$$\begin{cases} u_t(t) + \boldsymbol{A}u(t) + p(t)\boldsymbol{B}u(t) = 0, & t \in (0,\infty) \\ u(0) = u_0 \end{cases}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined by

$$D(\boldsymbol{A}) = H^2 \cap H_0^1(0,1), \;\; \boldsymbol{A}\varphi = -\frac{d^2\varphi}{dx^2} \quad \text{and} \quad \boldsymbol{B} \in \mathcal{L}(X), \;\; \boldsymbol{B}\varphi = \mu\varphi$$

- eigenvalues and eigenvectors of $\boldsymbol{A}$ are easily computed

$$\boldsymbol{\lambda_k} = (k\pi)^2, \quad \boldsymbol{\varphi_k}(x) = \sqrt{2}\sin(k\pi x) \quad \forall k \geq 1$$

Problem: to study local controllability to the ground state solution

Heat equation with Dirichlet boundary conditions

checking sufficient conditions

- gap condition

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k+1)\pi - k\pi = \pi \qquad \forall k \geq 1$$

Heat equation with Dirichlet boundary conditions

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$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k+1)\pi - k\pi = \pi \qquad \forall k \geq 1$$

- spreading condition

$$\langle \mu\varphi_1, \varphi_k \rangle = \sqrt{2}\int_0^1 \mu(x)\varphi_1(x)\sin(k\pi x)dx$$

$$= \sqrt{2}\left(-\left(\mu(x)\varphi_1(x)\right)\frac{\cos(k\pi x)}{k\pi}\Big|_0^1 + \int_0^1 (\mu(x)\varphi_1(x))'\frac{\cos(k\pi x)}{k\pi}dx \right)$$

$$= \sqrt{2}\left((\mu(x)\varphi_1(x))'\frac{\sin(k\pi x)}{(k\pi)^2}\Big|_0^1 - \int_0^1 (\mu(x)\varphi_1(x))''\frac{\sin(k\pi x)}{(k\pi)^2}dx \right)$$

$$= \sqrt{2}\left((\mu(x)\varphi_1(x))''\frac{\cos(k\pi x)}{(k\pi)^3}\Big|_0^1 - \int_0^1 (\mu(x)\varphi_1(x))'''\frac{\cos(k\pi x)}{(k\pi)^3}dx \right)$$

$$= \frac{4}{k^3\pi^2}\left[(-1)^{k+1}\mu'(1) - \mu'(0)\right] - \frac{\sqrt{2}}{(k\pi)^3}\int_0^1 (\mu(x)\varphi_1(x))'''\cos(k\pi x)dx$$

Heat equation with Dirichlet boundary conditions

checking speading condition

From the above computation it follows that

$$\langle B\varphi_1, \varphi_k\rangle = \frac{4}{k^3}\Big((-1)^{k+1}\mu'(1) - \mu'(0)\Big) - \frac{\sqrt{2}}{(k\pi)^3}\int_0^1 (\mu(x)\varphi_1(x))''' \cos(k\pi x)dx$$

So, if $\langle B\varphi_1, \varphi_k\rangle \neq 0 \ \forall k \in \mathbb{N}^*$ and $\mu'(1) \pm \mu'(0) \neq 0$, then we have

$$|\langle B\varphi_1, \varphi_k\rangle| \geq C\lambda_k^{-3/2}, \quad \forall k \in \mathbb{N}^*.$$

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Example For $B\varphi(x) = x^2\varphi(x)$ we have that

$$\langle B\varphi_1, \varphi_k\rangle = \begin{cases} \frac{(-1)^{k+1}4k}{(k^2-1)^2} & k \neq 1 \\ \frac{2\pi^2-3}{6\pi^2} & k = 1 \end{cases}$$

Heat equation with Dirichlet boundary conditions

local controllability to ground state solution

For any $T > 0$ there exists $R_T > 0$ such that for all u_0 with

$$\int_0^1 |u_0(x) - \sqrt{2}\sin(\pi x)|^2 dx < R_T^2$$

there exists $p \in L^2(0, T)$ steering the solution of the parabolic system

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)x^2 u(t,x) = 0 \quad (t,x) \in (0,T) \times (0,1) \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

to $u(T,x) = e^{-\pi^2 T}\sin(\pi x)$

Extensions

Unbounded control operators

Unbounded control operators

We allow for $B \notin \mathcal{L}(X)$ in $\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0 \quad (t > 0) \\ u(0) = u_0 \end{cases}$ $(\star)$

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Theorem

Suppose that $\lambda_1 \geq 0$ *and there exists a constant* $\alpha > 0$ *such that*

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \alpha \qquad \forall k \geq 1 \tag{G}$$

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$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \alpha \qquad \forall k \geq 1 \qquad (G)$$

Let $B : D(B) \subset X \to X$ *be a linear operator such that* $D(A^{1/2}) \subset D(B)$,

$$\|B\varphi\| \leq C\|\varphi\|_{D(A^{1/2})} \qquad \forall \varphi \in D(A^{1/2})$$

and

$$\langle B\varphi_1, \varphi_1 \rangle \neq 0 \qquad \& \qquad \exists\, \beta, q > 0 \quad \text{such that} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq \beta \quad \forall k > 1 \qquad (\star\star)$$

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Then for any $T > 0$ there exists $R_T > 0$ such that, $\forall u_0 \in D(A^{1/2})$ satisfying $\|A^{1/2}(u_0 - \varphi_1)\| < R_T$, the solution to $(\star)$ can be steered to the ground state solution in time T by some control $p \in L^2(0, T)$

Example (Fokker-Planck equation with Dirichlet b.c.)

Consider the bilinear control system

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)\big(\mu(x)u(t,x)\big)_x = 0, & (t,x) \in (0,\infty)\times(0,1) \\ u(t,0) = u(t,1) = 0 \quad \text{(absorbing b.c.)} \\ u(0,x) = u_0(x) \end{cases}$$

that translates into the evolution equation in $X = L^2(0,1)$

$$\begin{cases} u_t(t) + \color{magenta}{A}u(t) + p(t)\color{blue}{B}u(t) = 0, & t \in (0,\infty) \\ u(0) = u_0(x) \end{cases}$$

where $\color{magenta}{A}$ and $\color{blue}{B}$ are defined by

$$D(\color{magenta}{A}) = H^2 \cap H_0^1(0,1), \qquad \color{magenta}{A}\varphi = -\frac{d^2\varphi}{dx^2}$$

$$D(\color{blue}{B}) = \Big\{\varphi \in X \ : \ \frac{d}{dx}(\mu\varphi) \in X\Big\}, \quad \color{blue}{B}\varphi = \frac{d}{dx}(\mu\varphi)$$

Example (Fokker-Planck equation with Dirichlet b.c.)

checking assumptions

Observe that

- $D(A^{1/2}) = H_0^1(0,1) \subset D(B)$ if $\mu \in C^1([0,1])$
- the eigenvalues and eigenvectors of A are given by

$$\lambda_k = (k\pi)^2, \quad \varphi_k(x) = \sqrt{2}\sin(k\pi x) \quad \forall k \geq 1$$

- $\|B\varphi\| \leq \left(\|\mu\|_\infty^2 + \|\mu'\|_\infty^2\right)^{1/2} \|\varphi\|_{D(A^{1/2})}$ for any $\varphi \in D(A^{1/2})$

Remark

Since we have that

$$\int_0^1 \varphi_1(x)dx = \sqrt{2}\int_0^1 \sin(\pi x)dx = \frac{2\sqrt{2}}{\pi}$$

controlling the solution to the ground state means that we are forcing some mass to remain in the interval $[0,1]$ after time T (in the sense that with probability equal to $\frac{2\sqrt{2}}{\pi} \cong 0.9$ we find a particle in the interval $[0,1]$), even though we are in presence of absorbing boundary conditions

Example (Fokker-Planck equation with Dirichlet b.c.)

checking sufficient conditions

- gap condition $\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k+1)\pi - k\pi = \pi \qquad \forall k \geq 1$
- spreading condition

$$\begin{aligned}
\langle B\varphi_1, \varphi_k\rangle &= \sqrt{2}\int_0^1 (\mu\varphi_1)'(x)\sin(k\pi x)dx \\
&= \sqrt{2}\left(-(\mu\varphi_1)'(x)\frac{\cos(k\pi x)}{k\pi}\Big|_0^1 + \int_0^1 (\mu\varphi_1)''(x)\frac{\cos(k\pi x)}{k\pi}dx\right) \\
&= \frac{2}{k}\Big((-1)^k\mu(1)+\mu(0)\Big) + \frac{\sqrt{2}}{k\pi}\int_0^1 (\mu\varphi_1)''(x)\cos(k\pi x)dx.
\end{aligned}$$

If $\langle B\varphi_1, \varphi_k\rangle \neq 0 \ \forall k \in \mathbb{N}^*$ and $\mu(1) \pm \mu(0) \neq 0$, then we have

$$|\langle B\varphi_1, \varphi_k\rangle| \geq C{\lambda_k}^{-1/2}, \quad \forall k \geq 1$$

Example (Fokker-Planck equation with Dirichlet b.c.)

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- gap condition $\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k+1)\pi - k\pi = \pi \qquad \forall k \geq 1$
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If $\langle B\varphi_1, \varphi_k\rangle \neq 0\ \forall k \in \mathbb{N}^*$ and $\mu(1) \pm \mu(0) \neq 0$, then we have

$$|\langle B\varphi_1, \varphi_k\rangle| \geq C{\lambda_k}^{-1/2}, \quad \forall k \geq 1$$

EXAMPLE: $\boxed{B\varphi(x) = \frac{d}{dx}(x^n\varphi(x))}$ for any $n \geq 1$

Example (Fokker-Planck equation with Dirichlet b.c.)

local controllability

Remark

Taking $\mu(x) = x$ a direct check shows that the Fourier coefficients of $B\varphi_1$ do not vanish

$$\langle (\mu\varphi_1)', \varphi_k \rangle = \begin{cases} \frac{(-1)^k 2k}{k^2-1}, & k \geq 2 \\ \frac{1}{2} & k = 1 \end{cases}$$

and the lower bound $\lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq \beta$ is satisfied with $q = \frac{1}{2}$ and $\beta = 2\pi$

Our abstract result guarantees that, for any $n \geq 1$ and $T > 0$,

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)\big(xu(t,x)\big)_x = 0, & (t,x) \in (0,T) \times (0,1) \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

is locally exactly controllable to $\psi_1(T,x) = \sqrt{2}\, e^{-\pi^2 T} \sin(\pi x)$ by some control p

Global exact controllability on a strip

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0 \quad (t > 0) \\ u(0) = u_0 \end{cases} \qquad (S)$$

Theorem

Suppose that $\lambda_1 \geq 0$ *and there exists a constant* $\alpha > 0$ *such that*

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \alpha \qquad \forall k \geq 1$$

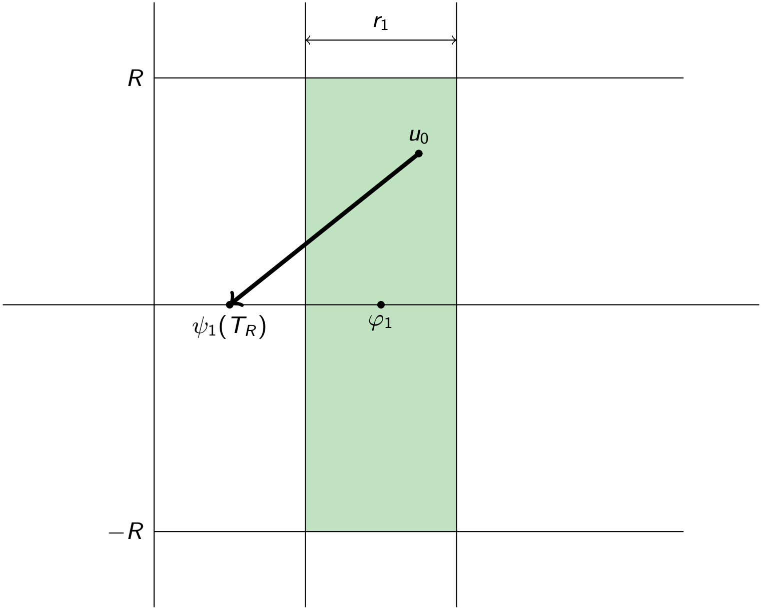
Let $B : X \to X$ *be a bounded linear operator satisfying the following*

$$\langle B\varphi_1, \varphi_1 \rangle \neq 0 \qquad \& \qquad \exists\, b, q > 0 \quad \text{such that} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1 \qquad (\star\star)$$

Then there exists $r_1 > 0$ *such that for all* $R > 0$ *there exists* $T_R > 0$ *such that for all* $u_0 \in X$ *in the strip*

$$\begin{aligned} &|\langle u_0, \varphi_1 \rangle - 1| \leq r_1 \\ &||u_0 - \langle u_0, \varphi_1 \rangle \varphi_1|| \leq R \end{aligned}$$

the solution to (S) *can be steered to the ground state solution* $\psi_1(t) = e^{-\lambda_1 t}\varphi_1$ *in time* T_R *by some control* $p \in L^2(0, T_R)$


r_1
R
u_0
$\psi_1(T_R)$
φ_1
$-R$

Global exact controllability outside $\varphi_1^\perp$

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0 & (t > 0) \\ u(0) = u_0 \end{cases} \qquad (S)$$

Corollary

Suppose that $\lambda_1 \geq 0$ *and there exists a constant* $\alpha > 0$ *such that*

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \alpha \qquad \forall k \geq 1$$

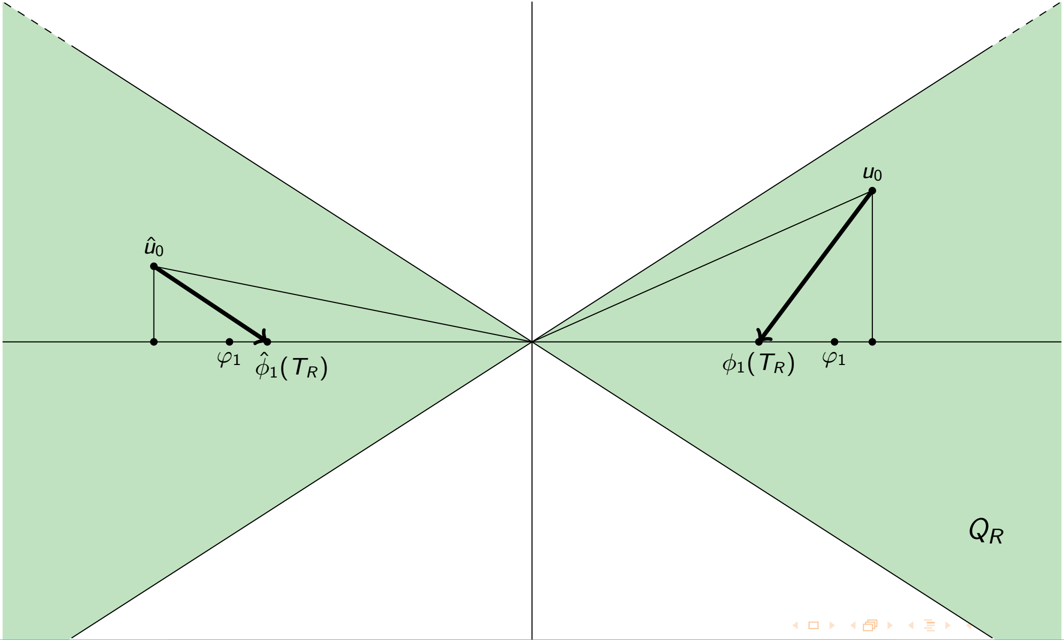
Let $B : X \to X$ *be a bounded linear operator satisfying the following*

$$\langle B\varphi_1, \varphi_1\rangle \neq 0 \qquad \& \qquad \exists\, b, q > 0 \quad \text{such that} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k\rangle| \geq b \quad \forall k > 1 \qquad (\star\star)$$

Then for every $R > 0$ *there exists* $T_R > 0$ *such that for all* u_0 *satisfying*

$$||u_0 - \langle u_0, \varphi_1\rangle\varphi_1|| \leq R|\langle u_0, \varphi_1\rangle|$$

the solution to (S) *can be steered to* $\langle u_0, \varphi_1\rangle\psi_1 =: \phi_1$ *in time* T_R *by some control* $p \in L^2(0, T_R)$



$\hat{u}_0$
φ_1
$\hat{\phi}_1(T_R)$
u_0
$\phi_1(T_R)$
φ_1
Q_R