Minicourse:
Inverse problems for time-fractional differential equations and classical diffusion equations

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Part I: Introduction

§1. Motivations to inverse problems

Purpose of Minicourse

- Inspiring to inverse problems for fractional differential equations
- Some aspects of recent research results (but not a survey)
- Explanations of available methods for inverse problems which I hope convenient for future researches

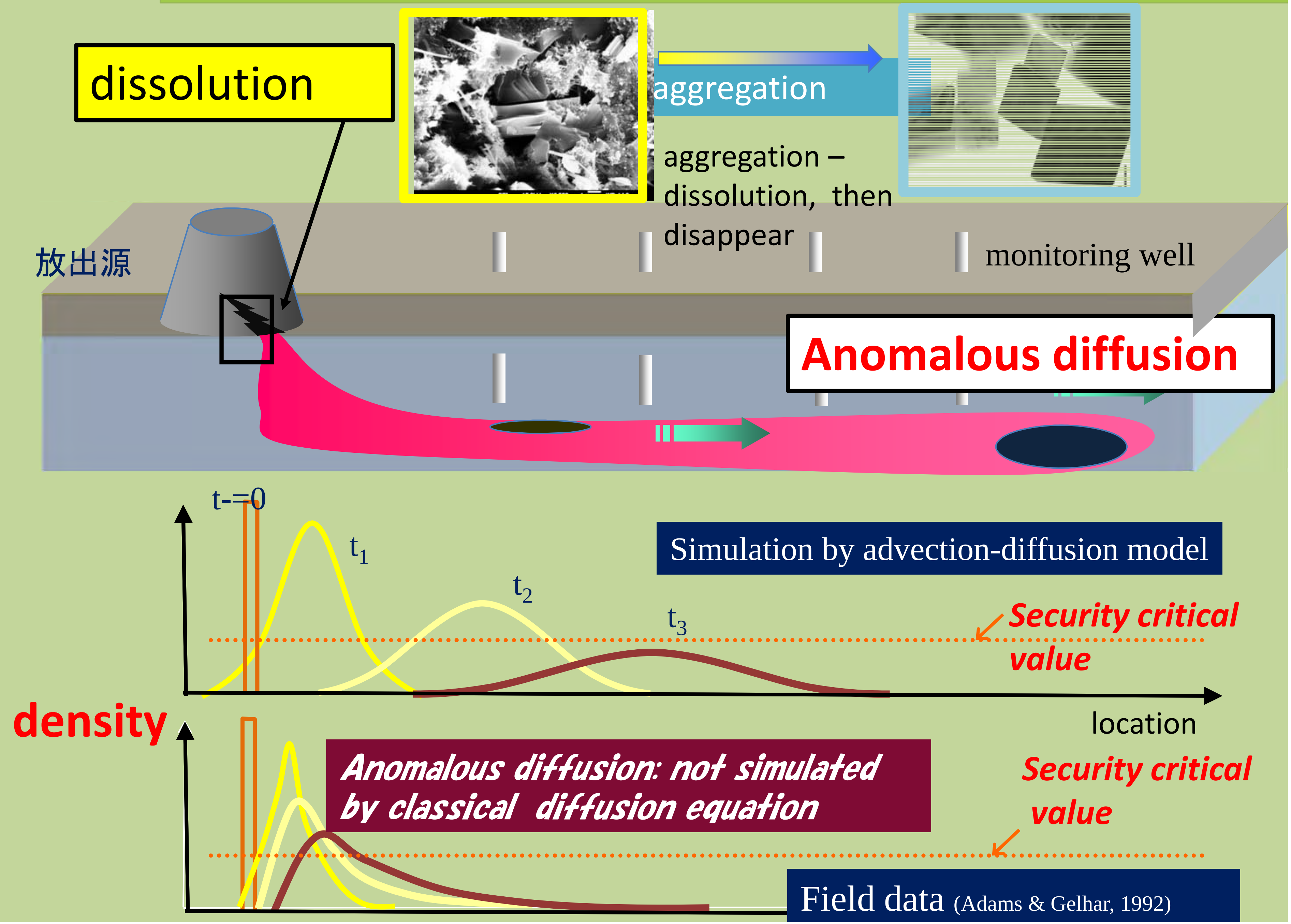
One motivation for inverse problems for time-fractional diffusion equations

For environmental issues, one has to consider
anomalous diffusion in heterogeneous media (e.g., soil)

Anomaly $\Longrightarrow$ Slow diffusion, long-tailed spatially profile of density
which cannot be described by classical advection-diffusion equations!

How can we model anomalous diffusion?
$\Longrightarrow$ One possibility is time-fractional diffusion-wave equation
as formulated later.

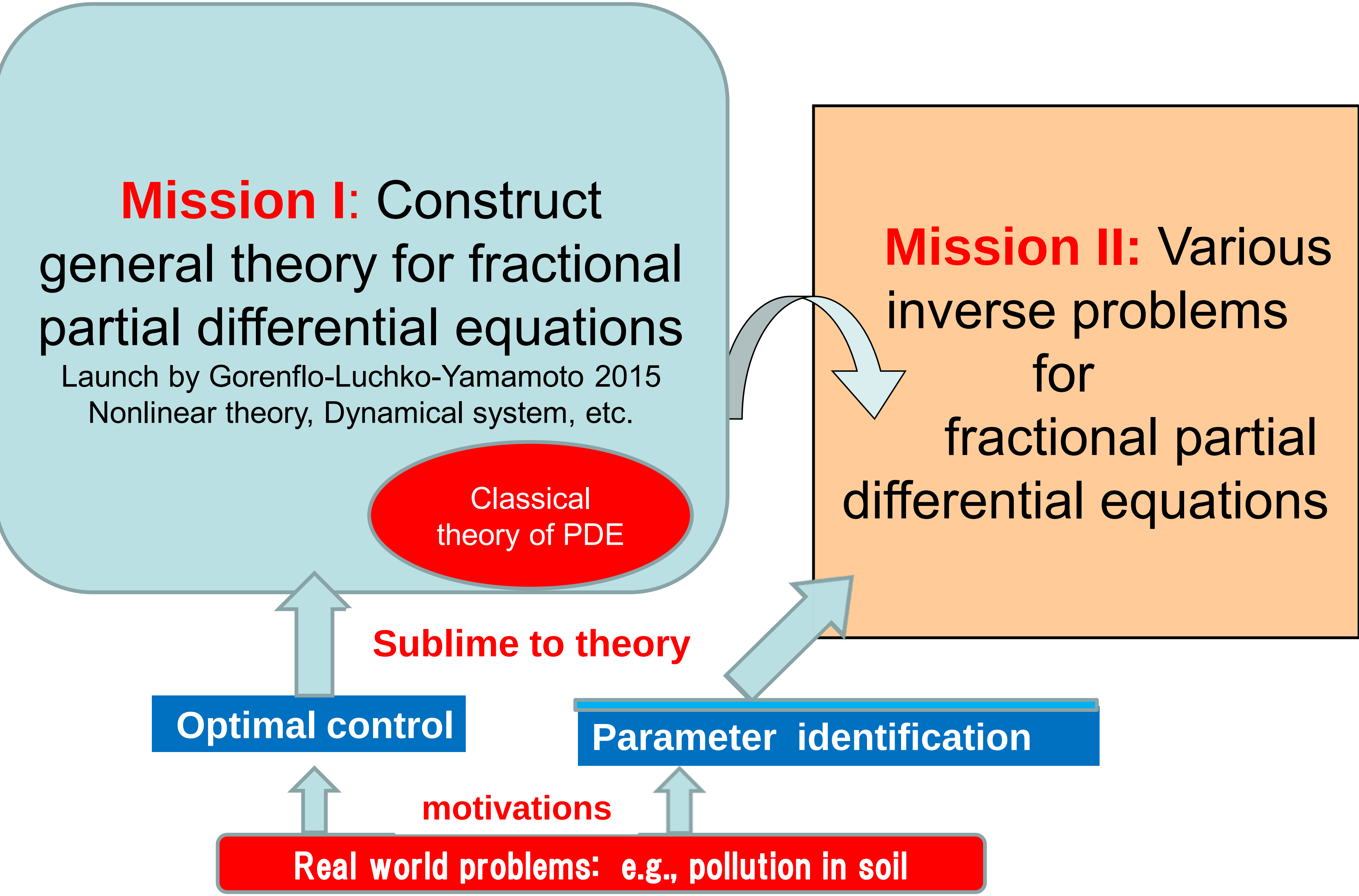
From ultraslow dissolution to anomalous diffusion



Motivations for inverse problems:

- How can we know a source of fatal contaminants?
  $\Longrightarrow$ inverse source problem.
- What are indices for characterizing the heterogeneity of media?
  $\Longrightarrow$ inverse problems of determining parameters.
- Identification of initial or past states by available a posterior data
  $\Longrightarrow$ Backward problem.

Grand Research Plan



Let $0 < \alpha < 1$.

- Dzherbashyan-Caputo derivative:

$$d_t^\alpha v(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} v'(s) ds \quad \text{for } v \in C^1[0,T] \text{ or } v \in W^{1,1}(0,T).$$

- Riemann-Liouville derivative:

$$D_t^\alpha v(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} v(s) ds \quad \text{for less regular } v.$$

Initial boundary value problem as direct problem

$$\begin{cases} d_t^\alpha u(x,t) = \Delta u + p(x)u + \mu(t) f(x), \quad x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = 0, \quad u(x,0) = a(x), \quad x \in \Omega, \qquad 0 < \alpha < 1. \end{cases}$$

We can consider more general fractional derivative:

$$\int_0^t (t-s)^{-\alpha} g(t-s) v'(s)\,ds, \quad \text{etc.}$$

with suitable g.

Determination of fractional derivatives

Inverse coefficient problems

$$\int_0^t (t-s)^{-\alpha} g(t-s)\partial_s u(x,s)ds = \mathrm{div}\,(p(x)\nabla u) + \mu(t)f(x)$$

$$u|_{\partial\Omega} = h$$

$$u(\cdot, 0) = a$$

Inverse heat conduction problem

Inverse source problems

Backward problem

Observability

Various inverse problems for fractional differential equations

Mathematical issues for inverse problems:

- uniqueness
- stability

Aspects of inverse problems:

- Very various inverse problems coming from ”a single” direct problem
  varieties of inverse problems × varieties
  of fractional equations as models
  $\Longrightarrow$ **Varieties**^{100}

  We are faced with many significant inverse problems for fractional differential equations!!

- How different from $\alpha = 1$ and $\alpha = 2$?

§2. Fractional calculus and fractional differential equations
Forward problems themselves are problematic!

d_t^α requires some regularity for $u(x,\cdot)$.
Such required regularity does not match with the expected regularity as solution to the forward problem within L^2-setting.

Example of initial value problem.

$$d_t^\alpha v(t) = t^\sigma \in L^2(0,T), \quad \sigma > -1/2, \qquad u(0) = a.$$

By classical solution formula, we can expect

$$v(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^\sigma ds = a + \frac{\Gamma(\sigma+1)}{\Gamma(\alpha+\sigma+1)} t^{\alpha+\sigma}$$

is the solution if $\alpha + \sigma > 0$.

$$v(t) = a + \frac{\Gamma(\sigma+1)}{\Gamma(\alpha+\sigma+1)} t^{\alpha+\sigma}.$$

However, if $0 < \alpha < \frac{1}{2}$, $-\frac{1}{2} < \sigma < 0$ and $\alpha + \sigma < 0$ (e.g., $\alpha = 1/4$, $\sigma = -3/8$), then the formula cannot give any solution.

Moreover $\alpha = 1/4$ and $\sigma = -1/4 \Longrightarrow v(t) =$ const. ????

$\Longrightarrow$ Formulation of d_t^α is needed for L^2-setting of initial boundary value problems.

First we make short descriptions of time-fractional derivatives in Sobolev spaces of fractional orders.

Let ${}_0C^1[0,T] := \{v \in C^1[0,T];\ v(0) = 0\}$ and $J^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds$ for $\alpha > 0$.

Theorem I-1.

Let $0 < \alpha < 1$

(i) $J^\alpha : L^2(0,T) \longrightarrow H_\alpha(0,T)$ *is isomorphism:*

$$H_\alpha(0,T) := \begin{cases} H^\alpha(0,T), & 0 < \alpha < 1/2, \\ \left\{v \in H^{1/2}(0,T);\ \int_0^T \frac{|v(t)|^2}{t} dt < \infty\right\}, & \alpha = 1/2, \\ \{v \in H^\alpha(0,T);\ v(0) = 0\}, & 1/2 < \alpha \le 1 \end{cases}$$

where

$$\|v\|_{H_\alpha(0,T)} := \begin{cases} \|v\|_{H^\alpha(0,T)}, & \alpha \ne 1/2, \\ \left(\|v\|^2_{H_{1/2}(0,T)} + \int_0^T \frac{|v(t)|^2}{t} dt\right)^{1/2}, & \alpha = 1/2. \end{cases}$$

Definition of ∂_t^α, $0 < \alpha \leq 1$:

$$\partial_t^\alpha := (J^\alpha)^{-1} = J^{-\alpha}$$

with

$$\mathcal{D}(\partial_t^\alpha) = H_\alpha(0,T) = J^\alpha L^2(0,T)$$

Then ∂_t^α is the minimum closed extension of $d_t^\alpha|_{{}_0C^1[0,T]}$.

Distinguish ∂_t^1 from $\frac{\partial}{\partial t}$ in $(\mathcal{D}(0,T))'$!

Remarks

- $\|\partial_t^\alpha v\|_{L^2(0,T)} \sim \|v\|_{H_\alpha(0,T)}$

- $v \in H_\alpha(0,T)$, $\alpha > 1/2 \Longrightarrow v(0) = 0$.
- $H_\alpha(0,T) \subsetneqq H^\alpha(0,T)$ for $\alpha \geq 1/2$.

- We can identify Dzherbashyan-Caputo derivative and Riemann-Liouville deriavtive with ∂_t^α in the domain $H_\alpha(0,T) = \partial_t^\alpha L^2(0,T)$: Moreover

$$\partial_t^\alpha v = D_t^\alpha v \quad \text{for all } v \in H_\alpha(0,T),$$

$$\partial_t^\alpha v = D_t^\alpha v = d_t^\alpha v \quad \text{for all } v \in {}_0C^1[0,T].$$

Definition of ∂_t^α with general $\alpha > 0$:

Let $\alpha := m + \sigma$, $m \in \mathbb{N}$, $0 < \sigma \le 1$

$$H_m(0,T) := \left\{ v \in H^m(0,T);\ v(0) = \cdots = \frac{d^{m-1}v}{dt^{m-1}}(0) = 0 \right\}$$

$$H_{m+\sigma}(0,T) := \left\{ v \in H_m(0,T);\ \frac{d^m v}{dt^m} \in H_\sigma(0,T) \right\}$$

Theorem I-2. $J^{m+\sigma} : L^2(0,T) \longrightarrow H_{m+\sigma}(0,T)$ *is isomorphism.*

Define $\partial_t^{m+\sigma} = J^{-m-\sigma}$ with $\mathcal{D}(\partial_t^{m+\sigma}) = H_{m+\sigma}(0,T)$

Theorem I-3. *(i)* $\partial_t^\alpha : H_{\alpha+\beta}(0,T) \longrightarrow H_\beta(0,T)$, *and* $J^\alpha : H_\beta(0,T) \longrightarrow H_{\alpha+\beta}(0,T)$ *are isomorphisms.*

(ii) $\partial_t^\alpha \partial_t^\beta = \partial_t^{\alpha+\beta}$ *in* $H_{\alpha+\beta}(0,T)$ *for* $\alpha, \beta > 0$

Fundamental theorems

Theorem I-4 (Laplace transform L for ∂_t^α with $\alpha > 0$).

$L(\partial_t^\alpha v)(p) = p^\alpha Lv(p)$ *for* $p > \exists C_v$ *if* $|v(t)| = O(e^{Ct})$ *as* $t \to \infty$ *and* $v|_{(0,T)} \in H_\alpha(0,T)$.

Theorem I-5 (adjoint).

$(\partial_t^\alpha)^* = ((J^\alpha)^*)^{-1}$ *with* $\mathcal{D}((\partial_t^\alpha)^*) = \tau H_\alpha(0,T)$ *for* $\alpha > 0$.

Here $(J^\alpha)^* f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s)ds = \tau J^\alpha(\tau f)(t)$

and $(\tau v)(t) := v(T-t)$.

Theorem I-6 (coercivity). *Let* $0 < \alpha < 1$. *Then* $\int_0^T v(t)\partial_t^\alpha v(t)dt \ge \frac{1}{2\Gamma(1-\alpha)} T^{-\alpha} \|v\|^2_{L^2(0,T)}$

$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\partial_s^\alpha v(s)ds \ge \frac{1}{2}|v(t)|^2$ *for all* $v \in H_\alpha(0,T)$

More extension of d_t^α to ${}^{-\beta}H(0,T)$.

Let $\beta > 0$ be arbitrary. We set

$$ {}^{\beta}H(0,T) := \{v(T-t); v \in H_\beta(0,T)\}. $$

Consider Gel'fand triple:

$$ {}^{\beta}H(0,T) \subset L^2(0,T) \subset {}^{-\beta}H(0,T) := ({}^{\beta}H(0,T))' \ : \text{ dual space.} $$

Then for any $\beta > 0$, we can extend ∂_t^α with $\mathcal{D}(\partial_t^\alpha) = {}^{-\beta}H(0,T)$.

Example.

$$ \partial_t^\alpha \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) = \delta \text{ (Dirac delta) in } {}^{-\beta}H(0,T) \text{ with } \beta > \frac{1}{2}. $$

Historial sketch.

- History of fractional calculus is very old:
  Leibniz mentioned in a letter to de l'Hôpital in 1695 and also to Bernouille brothers.
- N.H. Abel solved some mechanical problem in the article "Solution de quelques problèmes à l'aide d'intègrales definies" (1823), which is described by a solution of

$$J^{\frac{1}{2}} v(t) = f(t).$$

  with given f. He gives a solution $v(t) = D_t^{\frac{1}{2}} f(t)$.

  We can interpret that he considered $D_t^{\frac{1}{2}}$ from $J^{\frac{1}{2}}$.

  Our idea for ∂_t^α is similar: define ∂_t^α from J^α.
- However, fractional diffusion equations are much less old.

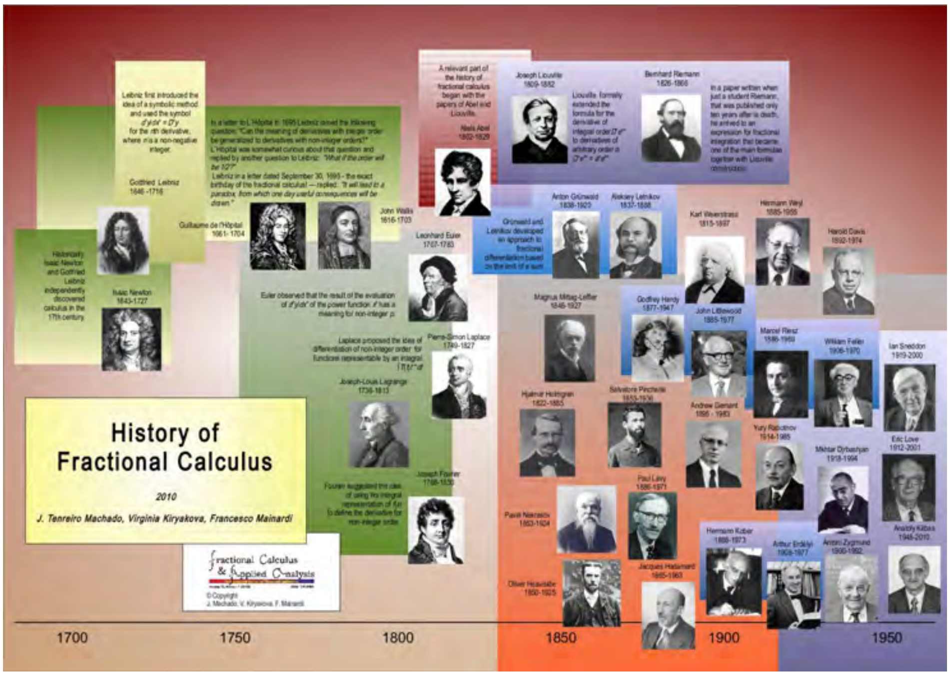


Figure 1: The time line of fractional calculus during the period 1695-1970

by J. Tenreiro Machado, V. Kiryakova, F. Mainardi

https://eudml.org/doc/219561

Basic results for Initial boundary value problem (IBVP).

Let $\Omega \subset \mathbb{R}^d$: bounded smooth domain, $0 < \alpha < 1$,

$-A(t)v(x) := \sum_{i,j=1}^d \partial_i(a_{ij}(x,t)\partial_j v) + \sum_{j=1}^d b_j(x,t)\partial_j v + c(x,t)v$ with

$a_{ij}, b_j, c \in C^2([0,T]; C^1(\overline{\Omega}))$ and uniform ellipticity.

Consider IBVP:

$$\begin{cases} \partial_t^\alpha (u-a) = -A(t)u + F(x,t), & x \in \Omega,\ 0 < t < T, \\ u(\cdot,t) \in H_0^1(\Omega), & 0 < t < T, \quad 0 < \alpha < 1, \\ (u-a)(x,\cdot) \in H_\alpha(0,T) \Longrightarrow & u(\cdot,0) = a \text{ if } \alpha > 1/2. \end{cases}$$

This coincides with classical formulation:

$$\begin{cases} d_t^\alpha u = -A(t)u + F(x,t), & x \in \Omega,\ 0 < t < T, \\ u|_{\partial\Omega} = 0, \quad u(x,0) = a(x), x \in \Omega, & \end{cases}$$

if u is suitably regular.

Basic well-posedness result:

Theorem I-7.

(i) $F \in L^2(0,T;H^{-1}(\Omega))$ *and* $a \in L^2(\Omega)$. *Then unique solution* $u \in L^2(0,T;H_0^1(\Omega))$ *exists such that* $u - a \in H_\alpha(0,T;H^{-1}(\Omega))$ *with a priori estimate.*

(ii) $F \in L^2(0,T;L^2(\Omega))$ *and* $a \in H_0^1(\Omega)$. *Then unique solution* $u \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ *exists such that* $u - a \in H_\alpha(0,T;L^2(\Omega))$ *with a priori estimate.*

Proof $\Longleftarrow$ Galerkin approximation method + coercivity of ∂_t^α

Ref: Kubica-Ryszewska-Y., Kubica-Y., Sakamoto-Y., Zacher, etc.
Within this framework we justify classical solution formula!

$$d_t^\alpha v(t) = f(t), \quad v(0) = a: \text{ what does it mean?}$$

and

$$\partial_t^\alpha (v(t) - a) = f(t), \quad v - a \in H_\alpha(0,T): \text{ well formulated}$$

are not same.

Formulation of IBVP for general $1 < \alpha < 2$

$\alpha = 1 + \sigma$ with $0 < \sigma \leq 1$:

$$\begin{cases} \partial_t^\alpha (u - a - bt) = -A(t)u + F(x,t), \quad x \in \Omega,\ 0 < t < T, \\ (u - a - bt)(x,\cdot) \in H_\alpha(0,T), \\ u(\cdot,t) \in H_0^1(\Omega), \quad 0 < t < T. \end{cases}$$

Remark.

Let $3/2 < \alpha < 2 \Longrightarrow$

$$(u - a - bt)(x,\cdot) \in H_\alpha(0,T) \quad \Longleftrightarrow$$

$$u(x,\cdot) = a, \quad \frac{\partial u}{\partial t}(\cdot,0) = b.$$

Solution formula for IBVP.

Let $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $c \in C^1(\overline{\Omega}), \le 0$. Let

$$(-Au)(x) = \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j u) + c(x)u(x), \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Let $0 < \lambda_1 < \lambda_2 < \cdots \quad \longrightarrow \infty$: set of eigenvalues of the operator A, and $\{\varphi_{nj}\}_{1\le j\le m_n}$ be an orthonormal basis in Ker $(A - \lambda_n)$. Set

$$P_n a := \sum_{j=1}^{m_n} (a, \varphi_{nj})\varphi_{nj}, \quad (\varphi,\ \psi) := \int_\Omega \varphi(x)\psi(x)dx.$$

Note that $A\varphi_{nj} = \lambda_n \varphi_{nj}$.

Mittag-Leffler functions

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \text{which is entire function in } z \in \mathbb{C}.$$

Let $a \in H_0^1(\Omega)$, $\mu \in L^2(0,T)$ and $f \in L^2(\Omega)$.

Solution formula.

(i) $0 < \alpha < 1$:

$$u(x,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha) P_n a(x) + \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) \mu(s) ds \right) P_n f(x)$$

in $L^2(0,T; H^2(\Omega))$ for $\partial_t^\alpha (u-a) + Au = \mu(t) f(x)$ and $u|_{\partial\Omega} = 0$.

(ii) $1 < \alpha < 2$:

$$u(x,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha) P_n a(x) + t E_{\alpha,\alpha}(-\lambda_n t^\alpha) P_n b(x)$$

$$+ \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) \mu(s) ds \right) P_n f(x)$$

in $L^2(0,T; H^2(\Omega))$ for $\partial_t^\alpha (u-a-bt) + Au = \mu(t) f(x)$ and $u|_{\partial\Omega} = 0$.

Part II. Inverse Problems

Contents

- Inverse problems require individual studies: each section is mutually independent.
- Some common techniques:
  1. Properties of Mittag-Leffler functions: Asymptotic expansions as $t \to \infty$, analyticity, complete monotonicity.
  2. Solution formula by eigenfunction expansion
  3. Laplace transform
- No integration by parts $\Longleftarrow$ no proper Carleman estimates
- Comparison with $\alpha = 1$ (heat equation) and $\alpha = 2$ (wave equation).

§1. Essence of available methods

Let $0 < \alpha < 1$ and

$$d_t^\alpha u - \Delta u = \mu(t) f(x), \quad u(x,0) = a(x), \quad u|_{\partial\Omega} = 0,$$

or

$$\partial_t^\alpha (u - a) - \Delta u = \mu(t) f(x), \qquad u|_{\partial\Omega} = 0, \quad (u-a)(x,\cdot) \in H_\alpha(0,T).$$

Let $A = -\Delta$ with $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

Let $0 < \lambda_1 < \lambda_2 < \cdots \longrightarrow \infty$: set of eigenvalues of the operator A, and $\{\varphi_{nj}\}_{1 \le j \le m_n}$ be an orthonormal basis in Ker $(A - \lambda_n)$. Set $P_n a := \sum_{j=1}^{m_j} (a, \varphi_{nj})\varphi_{nj}$.

Recall Mittag-Leffler functions:

$E_{\alpha,a}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}$ and $E_{\alpha,\alpha}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \alpha)}$.

Then

$$u(x,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^{\alpha}) P_n a(x) + \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^{\alpha}) \mu(s) ds \right) P_n f(x)$$

in $L^2(0,T;H^2(\Omega))$ and $u - a \in H_\alpha(0,T;L^2(\Omega))$ for given $a \in H_0^1(\Omega)$, $\mu \in L^2(0,T)$ and $f \in L^2(\Omega)$.

For simplicity, we assume that $a(x)$ and $f(x)$ are sufficiently smooth and suitably vanish near $\partial\Omega$.

Let $0 < \alpha < 1$.

(I): Determination of initial value a.

$$\partial_t^{\alpha}(u - a) = \Delta u, \quad u|_{\partial\Omega} = 0.$$

Let $\gamma \subset \partial\Omega$ be arbitrarily chosen subboundary and $0 < t_0$. Let $\partial_\nu u := \nabla u \cdot \nu$ and ν be outward normal vector to $\partial\Omega$.

Proposition 1.
If $\partial_\nu u = 0$ on $\gamma \times (0, t_0)$, then $a = 0$ in Ω.

Proof.

Step I. Let $\partial_\nu u = 0$ on $\gamma \times (0, t_0)$. Solution formula implies that $u(x,t)$ is analytic in $t > 0$ for fixed $x \in \Omega$. $\Longrightarrow \partial_\nu u = 0$ on $\gamma \times (0, \infty)$.

Step II. *Take Laplace transform:* $((Lv)(x,\cdot))(p) := \int_0^\infty e^{-pt} v(x,t)dt$.

Note that

$$L(E_{\alpha,1}(-\lambda_n t^\alpha))(p) = \frac{p^{\alpha-1}}{p^\alpha + \lambda_n} \quad \text{if } \mathrm{Re}\, p > \lambda_n^{\frac{1}{\alpha}}.$$

Hence analytic continuation implies

$$\sum_{n=1}^\infty \frac{\partial_\nu P_n a}{p^\alpha + \lambda_n} p^{\alpha-1} = 0 \quad \text{on } \gamma \text{ for } p > p_0\text{: some constant.}$$

Again analytic continuation implies

$$\sum_{n=1}^\infty \frac{\partial_\nu P_n a}{z + \lambda_n} = 0 \quad \text{on } \gamma \times (\mathbb{C} \setminus \{-\lambda_n\}_{n\in\mathbb{N}}).$$

Let Γ_n: small circle centered at $-\lambda_n$ excluding $-\lambda_m$ with $m \neq n$. Take $\int_{\Gamma_n} \cdots dz$.

Cauchy integral theorem and $\int_{\Gamma_n} \frac{dz}{z+\lambda_m} = 0$ imply $\partial_\nu P_n a = 0$ on γ for $n \in \mathbb{N}$.

$P_n a \in \mathcal{D}(A)$ implies $P_n a|_{\partial\Omega} = 0$.
$(\Delta - \lambda_n)P_n a = 0$ in Ω.

Unique continuation for elliptic equation $\Longrightarrow P_n a = 0$ in Ω for $n \in \mathbb{N}$.

$$a = \sum_{n=1}^{\infty} P_n a = 0 \quad \text{in } \Omega.$$

(II): Determination of source $f(x)$.

$$\partial_t^\alpha u = \Delta u + \mu(t) f(x), \quad u|_{\partial\Omega} = 0.$$

Let $\mu \in L^2(0,T)$, $\not\equiv 0$ in $(0, t_0)$.

Proposition 2.
If $\partial_\nu u = 0$ on $\gamma \times (0, t_0)$, then $f = 0$ in Ω.
Proof. Step I.

$$\partial_\nu u(x,t) = \int_0^t \sum_{n=1}^\infty (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) \mu(s) ds (\partial_\nu P_n f) = (G * \mu)(x,t) = 0 \quad 0 < t < t_0.$$

Here $(G * \mu)(x,t) := \int_0^t G(x, t-s)\mu(s) ds$ and $G(x,t) = \sum_{n=1}^\infty t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \partial_\nu P_n f$.

Then apply Titchmarsh theorem.

Titchmarsh theorem

Assme $g, \mu \in L^1(0, t_0)$ *and*

$$(g * \mu)(t) = 0, \quad 0 < t < t_0.$$

Then there exists $t_* \in (0,\ t_0)$ *such that*

$$g(t) = 0 \quad \textit{for } 0 < t < t_* \quad \textit{and} \quad \mu(t) = 0 \quad \textit{for } 0 < t < t_0 - t_*.$$

By $\mu \not\equiv 0$ in $(0, t_0)$, Titchmarsh theorem $\Longrightarrow$
$\exists\ t_* > 0$ such that $G(x, t) = 0$ for $0 < t < t_*$ and $x \in \gamma$.

$G(x, t)$ is analytic in $t > 0 \Longrightarrow G(x, t) = 0$ for $t > 0$ and $x \in \gamma$.

Step II.
Apply Laplace transform:

$$L(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha}))(p) = \frac{1}{p^{\alpha}+\lambda_n}, \quad \text{Re } p > \lambda_n^{\frac{1}{p}}$$

$\Longrightarrow$

$$(LG)(x,p) = \sum_{n=1}^{\infty} \frac{\partial_\nu P_n f}{p^{\alpha}+\lambda_n} = 0 \quad \text{on } \gamma \text{ for } t > 0.$$

Take $\int_{\Gamma_n} \cdots dz \Longrightarrow$

$$\partial_\nu P_n f = 0 \quad \text{on } \gamma \text{ for } n \in \mathbb{N}.$$

The same arguments yields $f = 0$ in Ω.

(III): Determination of derivative order $\alpha \in (0,1)$.
Let initial value a be fixed.

$$u_\alpha: \quad \partial_t^\alpha (u - a) = \Delta u, \quad u|_{\partial\Omega} = 0.$$

Proposition 3.
Let $a \ge 0$ or ≤ 0 in Ω and $a \not\equiv 0$, and $x_0 \in \Omega$ be arbitrary, $\alpha, \beta \in (0,1)$. Then $u_\alpha(x_0, t) = u_\beta(x_0, t)$ for $0 < t < t_0$ implies $\alpha = \beta$.
Proof.
Key: asymptotic expansion of Mittag-Leffler function:

$$E_{\alpha,1}(-\lambda_n t^\alpha) = \sum_{k=1, \alpha k \notin \mathbb{N}}^{N} \frac{(-1)^{k+1}}{\Gamma(1-\alpha k)} \frac{1}{\lambda_n^k t^{\alpha k}} + O\left(\frac{1}{t^{\alpha(N+1)}}\right) \quad \text{as } t \to \infty.$$

$u_\alpha(x,t), u_\beta(x,t)$ are analytic in $t > 0 \Longrightarrow$

$u_\alpha(x_0, t) = u_\beta(x_0, t)$ for $0 < t < t_0$ implies $u_\alpha(x_0, t) = u_\beta(x_0, t)$ for $t > 0 \Longrightarrow$

$$\sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha)(P_n a)(x_0) = \sum_{n=1}^{\infty} E_{\beta,1}(-\lambda_n t^\beta)(P_n a)(x_0), \quad t > 0$$

$\Longrightarrow$

$$\sum_{n=1}^{\infty} \left(\frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda_n t^\alpha} + O\left(\frac{1}{t^{2\alpha}} \right) \right) (P_n a)(x_0) = \sum_{n=1}^{\infty} \left(\frac{1}{\Gamma(1-\beta)} \frac{1}{\lambda_n t^\beta} + O\left(\frac{1}{t^{2\beta}} \right) \right) (P_n a)(x_0)$$

as $t \to \infty$.

Note that $\sum_{n=1}^{\infty} \frac{P_n a}{\lambda_n} = \sum_{n=1}^{\infty} A^{-1} P_n a = A^{-1} a$, where we set $A = -\Delta$.

$$\frac{1}{\Gamma(1-\alpha)}\frac{(A^{-1}a)(x_0)}{t^\alpha}+O\left(\frac{1}{t^{2\alpha}}\right)=\frac{1}{\Gamma(1-\beta)}\frac{(A^{-1}a)(x_0)}{t^\beta}+O\left(\frac{1}{t^{2\beta}}\right)$$

as $t \to \infty$.

Let $\alpha < \beta$. Then multiplying by t^α, we have

$$\frac{1}{\Gamma(1-\alpha)}(A^{-1}a)(x_0)+O\left(\frac{1}{t^{\alpha}}\right)=\frac{1}{\Gamma(1-\beta)}(A^{-1}a)(x_0)\frac{1}{t^{\beta-\alpha}}+O\left(\frac{1}{t^{2\beta-\alpha}}\right)$$

as $t \to \infty$. $\Longrightarrow$

$$\frac{1}{\Gamma(1-\alpha)}(A^{-1}a)(x_0) = 0$$

Set $b := A^{-1}a$. Then $Ab = -\Delta b = a \geq 0$ or ≤ 0 in Ω. Moreover $b(x_0) = 0$.
Strong and weak maximum principles imply $b = 0$ in Ω. Then $a = 0$ in Ω. This is contradiction and so $\alpha \geq \beta$. Similarly $\alpha \leq \beta$. Thus $\alpha = \beta$ is proved. ■

Supplements

Let $-\Delta b \geq 0$ *in* Ω *and* $b|_{\partial\Omega} = 0$. *If* $b(x_0) = 0$ *with some* $x_0 \in \Omega$, *then* $b = 0$ *in* Ω.

Proof.

By $-\Delta b \geq 0$ in Ω and $b|_{\partial\Omega} = 0$, the weak maximum principle implies

$$b \geq 0 \quad \text{in } \Omega.$$

Then $b(x_0) = 0$ means that $b(x)$ attains the minimim 0 at interior point $x_0 \in \Omega$.
Strong maximum principle implies that b is a constant function.
$b|_{\partial\Omega} = 0$ implies $b = 0$ in Ω. ■

§2. Determination of fractional orders (first visit)

$$u_{\alpha,\beta}\left\{\begin{array}{l} \partial_t^\alpha u = -(-\Delta)^\beta u, \quad x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = 0, \quad t > 0, \\ \left\{\begin{array}{ll} u(x,0) = a(x), \quad x \in \Omega & \text{if } 0 < \alpha < 1, \\ u(x,0) = a(x), \quad \partial_t u(x,0) = 0 & \text{if } 1 < \alpha < 2. \end{array}\right. \end{array}\right.$$

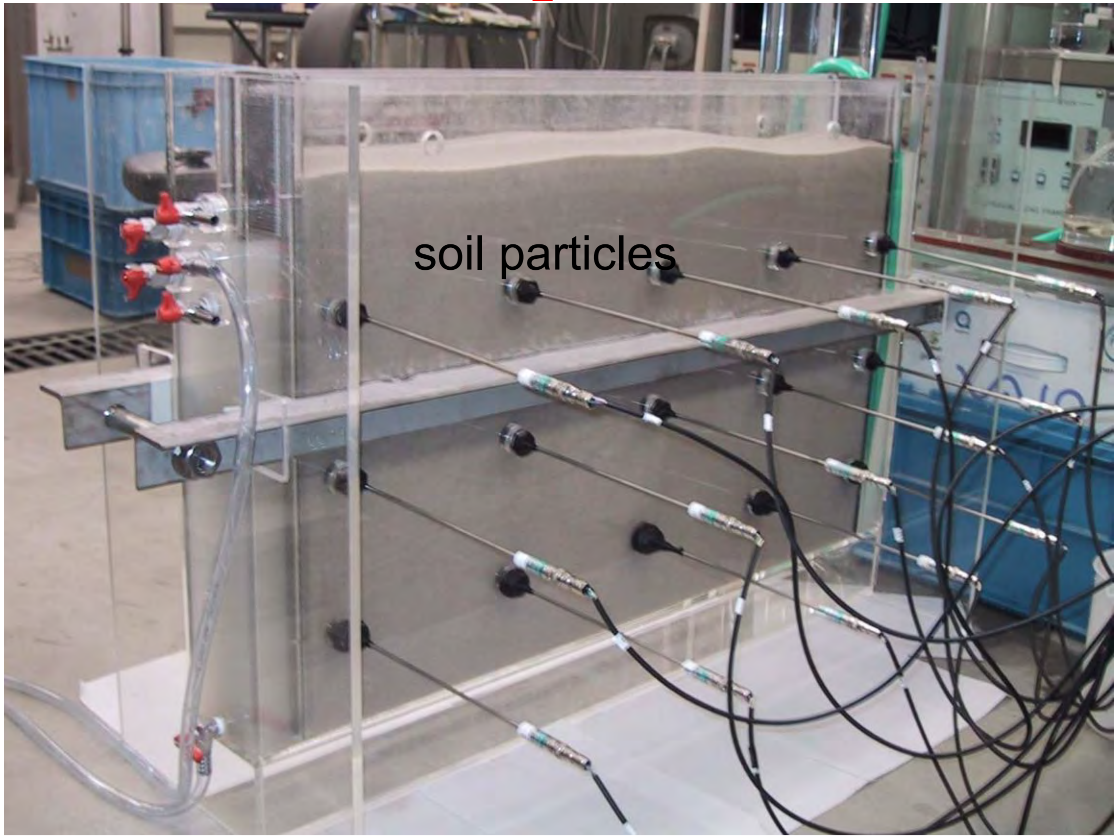
Inverse problem: determine $\alpha \in (0,2)$ and $\beta \in (0,1)$ by $u(x_0,t)$, $0 < t < T$ with fixed $x_0 \in \Omega$.

Remark. $(-\Delta)^{\beta}$: fractional power of $-\Delta$
$\Longleftarrow$ spatial derivative of order 2β
$\alpha, \beta \Longrightarrow$ important physical parameters

Inverse problem.
Let $x_0 \in \Omega$ be fixed. $u(x_0, t)$, $0 < t < T \Longrightarrow$
$\alpha \in ((0,2) \setminus \{1\})$ and $\beta \in (0,1)$.

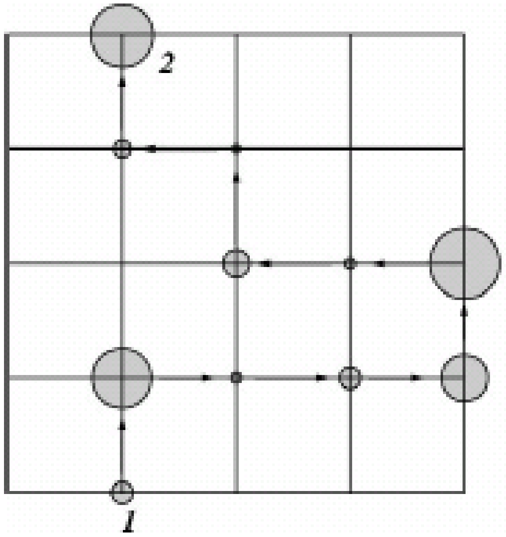
Uniqueness $\Longleftarrow$ How much information data have?

Determination of fractional orders at laboratory based on the micro-model



Prof. Y. Hatano (Tsukuba University)

Comparison of laboratory data with numerical results by Monte Carlo method



Ranom walk

Normal diffusion by Fick's law

$$\langle x^2 \rangle \propto t$$

α=1

Continuous time random walk (micro-model)

$$\langle x^2 \rangle \propto t^{\alpha} \qquad 0 < \alpha < 1$$

Related works (not comprehensive) on determination of orders

- Hatano, Nakagawa, Wang and Yamamoto 2013
- Li and Yamamoto 2015: uniqueness for mutiterm cases
- Yu, Jing and Qi 2015
- Janno 2016: unique existence
- Janno and Kinash 2018
- Krasnoschok, Pereverzyev, Siryk and Vasylyeva 2019
- Ashurov and Umarov 2020: arXiv:2005.13468v1

Other examples of data:

$$\int_{\Omega} u(x,t)\rho(x)dx, \quad 0 < t < T : \; \rho\text{: weight}$$

$$\nabla u \cdot \nu(x_0, t), \quad 0 < t < T$$

$\nu(x)$: unit outward normal vector

Preparations

$0 < \lambda_1 < \lambda_2 < \cdots \quad \longrightarrow \infty$: set of eigenvalues of $-\Delta$ with $u|_{\partial\Omega} = 0$.

$-\Delta\varphi_{nj} = \lambda_n \varphi_{nj}$, $1 \leq j \leq m_n$, m_n: multiplicity of λ_n

φ_{nj}, $1 \leq j \leq m_n$: orthonormal basis

$$(\varphi_{ni}, \varphi_{mj}) := \int_\Omega \varphi_{ni}(x)\varphi_{mj}(x)dx = \begin{cases} 1 & \text{if } n = m \text{ and } i = j, \\ 0 & \text{otherwise} \end{cases}$$

Define fractional powers of $-\Delta$:

$$(-\Delta)^{\beta} v := \sum_{n=1}^{\infty} \lambda_n^{\beta} \sum_{j=1}^{m_n} (v, \varphi_{nj}) \varphi_{nj},$$

$v \in D((-\Delta)^{\beta}) \subset H^{2\beta}(\Omega)$: Sobolev-Slobodeckij space

$$(-\Delta)^{-\beta} a = \frac{\sin \pi\beta}{\pi} \int_0^{\infty} \eta^{-\beta} (-\Delta + \eta)^{-1} a d\eta$$

in $L^2(\Omega)$ with $0 < \beta < 1$ (e.g., Pazy).

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} :$$ Mittag-Leffler function

Theorem 2.1 (uniqueness).

Let $a \in H_0^2(\Omega)$ *if* $d = 1, 2, 3$ *(*$a \in H_0^{2\sigma}(\Omega)$ *with* $\sigma > \frac{d}{4}$*),*

$$a \geq 0, \not\equiv 0 \quad \text{or} \quad \leq 0, \not\equiv 0 \quad \text{in } \Omega,$$

and

$$\exists n_0 \in \mathbb{N} \quad \text{such that } \lambda_{n_0} \neq 1 \text{ and } \sum_{j=1}^{m_{n_0}} (a, \varphi_{n_0 j})\varphi_{n_0 j}(x_0) \neq 0.$$

Then $u_{\alpha,\beta}(x_0, t)$*,* $0 < t < T$ *uniquely determines* $(\alpha, \beta) \in ((0, 2) \setminus \{1\}) \times (0, 1)$*.*

Simplified case.

Assume $m_n = 1$: λ_n is simple for all n, $-\Delta\varphi_n = \lambda_n\varphi_n$.

Corollary (uniqueness).

We assume (i) and (ii):

(i) $a \in H_0^2(\Omega)$ *if* $d = 1, 2, 3$,

$$a \geq 0, \not\equiv 0 \quad \text{or} \quad \leq 0, \not\equiv 0 \quad \text{in } \Omega$$

(ii) $\exists n_0 \in \mathbb{N}$, $(a, \varphi_{n_0})\varphi_{n_0}(x_0) \neq 0$ *and* $\lambda_{n_0} \neq 1$.

Then $u_{\alpha,\beta}(x_0, t)$, $0 < t < T \Longleftrightarrow (\alpha, \beta) \in ((0, 2) \setminus \{1\}) \times (0, 1)$ *1 to 1*

Remark.

- (ii) is essential for uniqueness of β.
  Let $\lambda_1 = 1$ and $-\Delta\varphi_1 = \varphi_1$. Then
  $u_{\alpha,\beta}(x,t) = E_{\alpha,1}(-t^\alpha)\varphi_1(x)$ for $x \in \Omega,\ t > 0$.
  No information of β!
- (i) $\Longrightarrow$ uniqueness for α
- Tatar-Ulusoy (2013):
  $(a, \varphi_n) > 0$ for all $n \Longrightarrow$ uniqueness.
  Our proof producces the same conclusion.
- Assume that $\lambda_n \neq 1$ for all n.
  If $a(x_0) \neq 0$, then the uniqueness for β holds.

Main ingredients for proof.
(1) Eigenfunction expansion
(2) Asymptotics of Mittag-Leflller function

$$E_{\alpha,1}(-t) = \sum_{\ell=1,\alpha\ell\notin\mathbb{N}}^{N} \frac{(-1)^{\ell+1}}{\Gamma(1-\alpha\ell)} \frac{1}{t^{\ell}} + O\left(\frac{1}{t^{1+N}}\right), \quad t > 0, \to \infty$$

(3) Strong maximum principle for $(-\Delta)^{\beta}$:

Lemma 2.1(strong maximum principle for $(-\Delta)^{-\beta}$):

$a \geq 0, \not\equiv 0, a|_{\partial\Omega} = 0 \Longrightarrow (-\Delta)^{-\beta} a(x) > 0, x \in \Omega.$

$\beta = 1 \Longrightarrow$ classical strong maximum principle

Proof:

We prove

$$(-\Delta + \eta)^{-1} a > 0 \quad \text{in } \Omega \text{ with all } \eta > 0.$$

Indeed, set $w := (-\Delta + \eta)^{-1} a \Longrightarrow$
$-\Delta w + \eta w = a$ in Ω. Then

$$\Delta w - \eta w = -a \leq 0 \quad \text{in } \Omega, \quad w|_{\partial\Omega} = 0.$$

Weak maxmimum principle implies $w \geq 0$ in Ω.
Then $\Delta w - \eta w = -a \leq 0$ in Ω and $w \geq 0$. Strong maximum principle and $\eta > 0$ imply that w cannot attain non-positive minimum in Ω if w is not constant. $\Longrightarrow$ If $\exists \widetilde{x} \in \Omega$ such that $w(\widetilde{x}) = 0$, then $w = 0$ in Ω. $\Longrightarrow a = 0$ in Ω, which is impossible.
Therefore $(-\Delta + \eta)a(x) = w(x) > 0$ for all $x \in \Omega$.

$(-\Delta + \eta)a = w > 0$ in Ω.

We have

$$((-\Delta)^{-\beta})a(x) = \frac{\sin \pi\beta}{\pi} \int_0^\infty \eta^{-\beta}(-\Delta + \eta)^{-1} a \, d\eta$$

(e.g., Pazy).

Therefore $((-\Delta)^{-\beta})a(x) > 0$ for all $x \in \Omega$. Lemma 2.1 is proved. ■

Proof for simple eigenvalues

$$u_{\alpha,\beta}(x_0,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n^{\beta} t^{\alpha})(a,\varphi_n)\varphi_n(x_0)$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{1}{t^{\alpha}} \sum_{n=1}^{\infty} \frac{(a,\varphi_n)\varphi_n(x_0)}{\lambda_n^{\beta}} + O\left(\frac{1}{t^{2\alpha}}\right)$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{1}{t^{\alpha}} (-\Delta)^{\beta} a(x_0) + O\left(\frac{1}{t^{2\alpha}}\right) \quad \text{as } t \to \infty$$

Note that

$$\frac{(a,\varphi_n)\varphi_n(x_0)}{\lambda_n^{\beta}} = (a,\varphi_n)((-\Delta)^{-\beta}\varphi_n)(x_0)$$

and

$$\sum_{n=1}^{\infty} \frac{(a,\varphi_n)\varphi_n(x_0)}{\lambda_n^{\beta}} = \sum_{n=1}^{\infty} (a,\varphi_n)((-\Delta)^{-\beta}\varphi_n)(x_0) = (-\Delta)^{-\beta} a(x_0).$$

$$u_{\alpha,\beta}(x_0,t) = u_{\alpha_1,\beta_1}(x_0,t),\ 0 < t < T \Longrightarrow$$

$$\frac{1}{\Gamma(1-\alpha)}\frac{1}{t^{\alpha}}(-\Delta)^{-\beta}a(x_0) + O\left(\frac{1}{t^{2\alpha}}\right) = \frac{1}{\Gamma(1-\alpha_1)}\frac{1}{t^{\alpha_1}}(-\Delta)^{-\beta_1}a(x_0) + O\left(\frac{1}{t^{2\alpha_1}}\right)$$

as $t \to \infty \Longrightarrow$

$(-\Delta)^{-\beta}a(x_0) \neq 0$, $(-\Delta)^{-\beta_1}a(x_0) \neq 0$ by $x_0 \in \Omega \Longrightarrow \alpha = \alpha_1$

Proof of $\beta = \beta_1$.

Set $a_n = (a, \varphi_n)\varphi_n(x_0)$.

$$\sum_{\ell=1, \alpha\ell\notin\mathbb{N}}^{N} \frac{(-1)^{\ell+1}}{\Gamma(1-\alpha\ell)t^{\alpha\ell}} \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^{\beta\ell}} = \sum_{\ell=1, \alpha\ell\notin\mathbb{N}}^{N} \frac{(-1)^{\ell+1}}{\Gamma(1-\alpha\ell)t^{\alpha\ell}} \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^{\beta_1\ell}} + O\left(\frac{1}{t^{\alpha(1+N)}}\right)$$

as $t \to \infty \Longrightarrow$

$$\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^{\beta\ell}} = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^{\beta_1\ell}} \quad \text{for } \ell \in \mathbb{N} \setminus \left\{\frac{m}{\alpha}\right\}_{m\in\mathbb{N}}.$$

Let $n_0 \in \mathbb{N}$ such that $a_1 = \cdots = a_{n_0 - 1} = 0$ and $a_{n_0} \neq 0$, $\lambda_{n_0} \neq 1$.
Assume $\lambda_{n_0} > 1$ (similar for other case): Let $\beta_1 > \beta \Longrightarrow$

$$a_{n_0}\left(1 - \frac{\lambda_{n_0}^{\beta\ell}}{\lambda_{n_0}^{\beta_1\ell}}\right) + \sum_{n=n_0+1}^{\infty} a_n\left(\left(\frac{\lambda_{n_0}^{\beta}}{\lambda_n^{\beta}}\right)^{\ell} - \left(\frac{\lambda_{n_0}^{\beta}}{\lambda_n^{\beta_1}}\right)^{\ell}\right) = 0$$

for $\ell \in \mathbb{N} \setminus \{m/\alpha\}_{m \in \mathbb{N}}$.

$$\left|\frac{\lambda_{n_0}^{\beta}}{\lambda_{n_0}^{\beta_1}}\right|, \quad \left|\frac{\lambda_{n_0}^{\beta}}{\lambda_n^{\beta}}\right|, \quad \left|\frac{\lambda_{n_0}^{\beta}}{\lambda_n^{\beta_1}}\right| < 1$$

for $n \geq n_0 + 1$.
Choose $\ell_k \in \mathbb{N} \setminus \{m/\alpha\}_{m \in \mathbb{N}}$ such that $\lim_{k \to \infty} \ell_k = \infty$
$\Longrightarrow a_{n_0} = 0$: contradiction. Hence $\beta = \beta_1$. ■

§3. Determination of fractional orders (revisit)

$$u_{\alpha,a,b}\begin{cases} \partial_t^\alpha (u-a) = -Au, \quad x\in\Omega,\ t>0, \quad \text{if } 0<\alpha\le 1, \\ \partial_t^\alpha (u-a-bt) = -Au, \quad x\in\Omega,\ t>0, \quad \text{if } 1<\alpha<2, \\ u(\cdot,t)\in H_0^1(\Omega), \qquad t>0. \end{cases}$$

Let $-Av(x) = \sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j v) + c(x)v$ with smooth a_{ij}, c and $c\le 0$ in Ω, $\Omega\subset\mathbb{R}^d$ with $d=1,2,3$, and $a,b\in\mathcal{D}(A^\sigma)$ with $\sigma>\frac{d}{4}$ (OK if $a,b\in H_0^2(\Omega)$).

Inverse problem. Let $x_0\in\Omega$ be fixed.
$u_{\alpha,a,b}(x_0,t)$, $0<t<T \Longrightarrow \alpha\in(0,2)$.

Remark: d (spatial dim.) ≤ 3 and $a,b\in\mathcal{D}(A^\sigma) \Longrightarrow u_{\alpha,a,b}(\cdot,t)\in C(\overline{\Omega})$. Therefore data are well-posed.

Theorem 3.1 (uniqueness). *Let* $0 < \alpha, \beta < 2$. *Let* $b = \widetilde{b} = 0$ *if* $1 < \alpha, \beta < 2$. *Assume*

$$a \ge 0, \not\equiv 0 \quad \text{or} \quad \le 0, \not\equiv 0 \quad \text{in } \Omega.$$

Then $u_{\alpha,a,0}(x_0, t) = u_{\beta,\widetilde{a},0}(x_0, t)$, $0 < t < T$
$\Longleftrightarrow \alpha = \beta$.

We can prove the uniqueness for more general initial values.
Theorem 3.2 (uniqueness).
Let either hold:

$$a(x_0) \ne 0 \quad \text{or} \quad \widetilde{a}(x_0) \ne 0 \quad \text{or} \quad b(x_0) \ne 0 \quad \text{or} \quad \widetilde{b}(x_0) \ne 0.$$

Then $u_{\alpha,a,b}(x_0, t) = u_{\beta,\widetilde{a},\widetilde{b}}(x_0, t)$, $0 < t < T \Longrightarrow \alpha = \beta$.

The order α is very distinguished index for the equation: $\Longrightarrow$
α is determined independently even for unknown elliptic part and initial values!

Let $0 < \alpha, \beta < 2$, $\gamma \subset \partial\Omega$; any subboundary, and $A, \widetilde{A}$ be regular symmetric elliptic operators with time independent coefficients and $u_{\alpha,a,b,A}(x,t)$ satisfy

$$\begin{cases} \partial_t^\alpha (u - a) = -Au & \text{if } 0 < \alpha \le 1, \\ \partial_t^\alpha (u - a - bt) = -Au & \text{if } 1 < \alpha < 2, \\ & u(\cdot, t) \in H_0^1(\Omega), \end{cases}$$

Theorem 3.3.

If

$$\nabla u_{\alpha,a,b,A} = \nabla u_{\beta,\widetilde{a},\widetilde{b},\widetilde{A}} \quad \text{on } \gamma \times (0,T),$$

then either $\alpha = \beta$ *or* $u = v = 0$ *in* $\Omega \times (0,T)$.

Recall:

$0 < \lambda_1 < \lambda_2 < \cdots \longrightarrow \infty$: set of eigenvalues of $-\Delta$ with $u|_{\partial\Omega} = 0$.

Let $\{\varphi_{nj}\}_{1 \le j \le m_n}$ be an orthonomal basis of Ker $(A - \lambda_n)$.

Mittag-Leffler functions

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 2)}$$

Main ingredients for proof.
(1) Eigenfunction expansion
(2) Asymptotics of Mittag-Lefller function

$$E_{\alpha,1}(-t) = \sum_{\ell=1,\alpha\ell\notin\mathbb{N}}^{N} \frac{(-1)^{\ell+1}}{\Gamma(1-\alpha\ell)} \frac{1}{t^{\ell}} + O\left(\frac{1}{t^{1+N}}\right), \quad t>0, \to \infty$$

$$E_{\alpha,\alpha}(-t) = \sum_{\ell=1,\alpha\ell\notin\mathbb{N}}^{N} \frac{(-1)^{\ell+1}}{\Gamma(2-\alpha\ell)} \frac{1}{t^{\ell}} + O\left(\frac{1}{t^{1+N}}\right), \quad t>0, \to \infty$$

Remark: Note $\frac{1}{\Gamma(\gamma)} = 0$ for $\gamma = 0, -1, -2, \ldots$

Proof. Let $\alpha \geq \beta$.
For simplicity let initial values be smooth and λ_n be simple.

$$u_{\alpha,a,b}(x,t) = \sum_{n=1}^{\infty} \{(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^{\alpha}) + (b,\varphi_n)tE_{\alpha,\alpha}(-\lambda_n t^{\alpha})\}\varphi_n(x)$$

+ asymptoptics of $E_{\alpha,1}$ and $E_{\alpha,\alpha}$ $\Longrightarrow$

$$\sum_{\ell=1,\alpha\ell\notin\mathbb{N}}^{N} \frac{(-1)^{\ell}}{\Gamma(1-\alpha\ell)} \frac{1}{t^{\alpha\ell}} \sum_{n=1}^{\infty} \frac{p_n}{\lambda_n^{\ell}} + \sum_{\ell=1,\alpha\ell\notin\mathbb{N}}^{N} \frac{(-1)^{\ell}}{\Gamma(2-\alpha\ell)} \frac{1}{t^{\alpha\ell-1}} \sum_{n=1}^{\infty} \frac{q_n}{\lambda_n^{\ell}}$$

$$= \sum_{\ell=1,\alpha\ell\notin\mathbb{N}}^{N} \frac{(-1)^{\ell}}{\Gamma(1-\beta\ell)} \frac{1}{t^{\beta\ell}} \sum_{n=1}^{\infty} \frac{\widetilde{p}_n}{\lambda_n^{\ell}} + \sum_{\ell=1,\alpha\ell\notin\mathbb{N}}^{N} \frac{(-1)^{\ell}}{\Gamma(2-\beta\ell)} \frac{1}{t^{\beta\ell-1}} \sum_{n=1}^{\infty} \frac{\widetilde{q}_n}{\lambda_n^{\ell}} + O\left(\frac{1}{t^{\beta N+\beta-1}}\right)$$

Here $p_n = (a,\varphi_n)\varphi_n(x_0)$, $q_n = (b,\varphi_n)\varphi_n(x_0)$, $\widetilde{p}_n = (\widetilde{a},\varphi_n)\varphi_n(x_0)$, $\widetilde{q}_n = (\widetilde{b},\varphi_n)\varphi_n(x_0)$

Remark that $d \leq 3 \Longrightarrow$

$$\sum_{n=1}^{\infty} \frac{\widetilde{p_n}}{\lambda_n} < \infty, \text{ etc.}$$

Key of Proof is separation of powers in asymptotics:

For $k \in \mathbb{N}$, find $\ell_k \in \mathbb{N}$ such that $\lim_{k\to\infty} \ell_k = \infty$,

$$\beta\ell_k \notin \mathbb{N} \cup \{\beta m - 1\}_{m\in\mathbb{N}} \cup \{\alpha m\}_{m\in\mathbb{N}} \cup \{\alpha m - 1\}_{m\in\mathbb{N}}$$

$\Longrightarrow$ Terms of $\frac{1}{t^{\beta\ell_k}}$ appear in the series only one time as term

$$\frac{(-1)^{\ell_k}}{\Gamma(1-\beta\ell_k)} \sum_{n=1}^{\infty} \frac{\widetilde{p_n}}{\lambda_n^{\ell_k}} \frac{1}{t^{\beta\ell_k}}.$$

Then we can verify $\sum_{n=1}^{\infty} \frac{\widetilde{p_n}}{\lambda_n^{\ell_k}} = 0, \quad \forall k \in \mathbb{N}$

$\sum_{n=1}^{\infty} \frac{\widetilde{p}_n}{\lambda_n^{\ell_k}} = 0$ for all $k \in \mathbb{N}$ implies $\widetilde{p}_n = (\widetilde{a}, \varphi_n)\varphi_n(x_0) = 0$ for all $n \in \mathbb{N}$ $\Longleftarrow$

Lemma 3.1.

Let $\sum_{n=1}^{\infty} \frac{\widetilde{p}_n}{\lambda_n} < \infty$, $k \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{\infty} \frac{\widetilde{p}_n}{\lambda_n^{\ell_k}} = 0, \quad k \in \mathbb{N} \quad \Longrightarrow \quad \widetilde{p}_n = 0, \quad n \in \mathbb{N} \qquad \blacksquare$$

Then $p_n = q_n = \widetilde{q}_n = 0$ for all $n \in N$ follow.

Proof of Lemma 3.1. $\frac{\widetilde{p}_1}{\lambda_1^{\ell_k}} + \sum_{n=2}^{\infty} \frac{\widetilde{p}_n}{\lambda_n^{\ell_k}} = 0 \Longrightarrow \widetilde{p}_1 + \sum_{n=2}^{\infty} \widetilde{p}_n \left(\frac{\lambda_1}{\lambda_n}\right)^{\ell_k} = 0$

By $\left|\frac{\lambda_1}{\lambda_n}\right| < 1$, $k \to \infty \Longrightarrow \widetilde{p}_1 = 0$. Repeat and $\widetilde{p}_1 = \widetilde{p}_2 = \cdots = 0$.

Main issue for proof: how to find such sequence $\ell_k,\ k \in \mathbb{N}$

trivial if $\alpha, \beta \notin \mathbb{Q}$ and $\frac{\alpha}{\beta} \notin \mathbb{Q}$

because

$$\beta\ell \notin \mathbb{N} \cup \{\beta m - 1\}_{m\in\mathbb{N}} \cup \{\alpha m\}_{m\in\mathbb{N}} \cup \{\alpha m - 1\}_{m\in\mathbb{N}}$$

for any $\ell \in \mathbb{N}$.

In general if $\alpha, \beta \in \mathbb{Q}$, then more consideration is necessary by residue class of integers such as Diophantine equation.

§4. Initial values and decay rates

Let $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^d$ bounded domain with $d = 1, 2, 3$ for simplicity:

$$\partial_t^\alpha (u - a) = -A(t)u, \quad x \in \Omega,\ t > 0, \quad u|_{\partial\Omega} = 0.$$

Here $-A(t)v = \sum_{i,j=1}^d \partial_i(a_{ij}(x,t)\partial_j v) + c(x,t)v$ with $c \le 0, \in L^\infty(\Omega \times (0,\infty))$ and $a_{ij} \in C^1([0,\infty); C(\overline{\Omega}))$.

Theorem 0 (Vergara-Zacher 2015, Kubica-Ryszewska-Y. 2020).

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le \frac{C\|a\|_{L^2(\Omega)}}{t^\alpha}$$

Question: $\frac{1}{t^\alpha}$ is best possible decay rate?

Proposition 0 (Sakamoto-Yamamoto 2011).
Let $\omega \subset \Omega$*: some subdomain,* $u(\cdot, 0)$ *be smooth.*
Then: $\forall m \in \mathbb{N}$, $\exists C_m > 0$ *such that*

$$\|u(\cdot, t)\|_{L^\infty(\omega)} \le \frac{C_m}{t^m}$$

$\Longrightarrow u = 0$ *in* $\Omega \times (0, T)$.

$\Longleftarrow$ We improve this result.

Let $-Av := \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j v) + c(x,t)v$ with $c \le 0$, and $\omega \subset \Omega$ be subdomain and $x_0 \in \Omega$.

Theorem 4.1.

(i) $\|u(\cdot,t)\|_{L^2(\omega)} = o\left(\frac{1}{t^\alpha}\right)$ *as* $t \to \infty \Longrightarrow u(\cdot,0)|_\omega = 0$.

(ii) Let $a \in H^\varepsilon(\Omega)$ *with some* $\varepsilon > 0$ *and* $\partial\omega \cap \partial\Omega$ *be open. Then*

$\|u(\cdot,t)\|_{L^2(\omega)} = o\left(\frac{1}{t^\alpha}\right)$ *as* $t \to \infty \Longleftrightarrow u(\cdot,0)|_\omega = 0$.

(iii) Let a *be sufficiently smooth. Then* $|Au(x_0,t)| = o\left(\frac{1}{t^\alpha}\right)$ *as* $t \to \infty \Longleftrightarrow u(x_0,0) = 0$.

Remark 1.
Initial value greatly influences asymptotic behavior ponitwise $\Longleftarrow$ anomalous diffusion
Very different from $\alpha = 1$

$$\begin{cases} \partial_t u = \partial_x^2 u(x,t), \quad 0 < x < \pi,\ t > 0, \\ u(0,t) = u(\pi,t) = 0, \quad u(x,0) = a(x) \end{cases}$$

$\Longrightarrow |u(x,t)| = O(e^{-t})$ in general. If $\int_0^\pi a(x) \sin x dx = 0 \Longrightarrow |u(x_0,t)| = O(e^{-4t})$.

But for $\alpha = 1$, data do not keep more specific information of a because of much faster decay!
$\Longrightarrow$ Decay rate does not provide no information of zeros of initial value
$\Longleftarrow$ strong smoothing

Remark 2. Same for $1 < \alpha < 2$.

Remark 3: nonsymmetric case:

Let

$$-Av(x) := \sum_{i,j=1}^{d} \partial_i (a_{ij}(x) \partial_j v) + \sum_{j=1}^{d} b_j(x) \partial_j v + c(x,t)v,$$

where $c \leq 0$ and $\inf_{x\in\Omega} |c(x)|$ is large such that real parts of the spectra of A are positive.

We can prove Theorem 4.1 for non-symmetric A.

Sketch of Proof of Theorem 4.1.
Key (same for §3)

- Eigenfunction expansion
- Asymptotic behavior:

$$E_{\alpha,1}(-\lambda t^{\alpha}) = \frac{1}{\Gamma(1-\alpha)}\frac{1}{\lambda t^{\alpha}} + O\left(\frac{1}{\lambda^2 t^{2\alpha}}\right) \quad \text{as } t \to \infty.$$

No such asymptotics for

$$e^{-\lambda t} := E_{1,1}(-\lambda t)!$$

Proof of (i). For simplicity, assume that all the eigenvalues are simple.
$u(x,t) = \sum_{n=1}^{\infty}(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x)$ + Asymptotics $\Longrightarrow$

$$u(x,t) = \frac{1}{\Gamma(1-\alpha)t^\alpha}\sum_{n=1}^{\infty}\frac{(a,\varphi_n)\varphi_n(x)}{\lambda_n} + O\left(\frac{1}{t^{2\alpha}}\right) = \frac{1}{\Gamma(1-\alpha)}(A^{-1}a)(x)\frac{1}{t^\alpha} + O\left(\frac{1}{t^{2\alpha}}\right)$$

$\Longrightarrow \|u(\cdot,t)\|_{L^2(\omega)} = \frac{1}{\Gamma(1-\alpha)}\frac{1}{t^\alpha}\|A^{-1}a\|_{L^2(\omega)} + o\left(\frac{1}{t^\alpha}\right) = o\left(\frac{1}{t^\alpha}\right)$
$\Longrightarrow f := A^{-1}a$ in Ω and $f = 0$ in $\omega \Longrightarrow a = Af = 0$ in ω.

Proof of (iii).
$Au(x,t) = \frac{-a(x)}{\Gamma(1-\alpha)t^\alpha} + O\left(\frac{1}{t^{2\alpha}}\right) \Longrightarrow$

$|Au(x_0,t)| = o\left(\frac{1}{t^\alpha}\right) \Longleftrightarrow a(x_0) = 0.$

§5. Fractional diffusion equations backward in time for $0 < \alpha < 1$

Based on a joint work with Profs G. Floridia (Università Mediterranea di Reggio Calabria) and Z. Li (Shandong University of Technology)

$-Au := \sum_{i,j=1}^{n} \partial_i (a_{ij}(x) \partial_j u) + c(x)u$,

where $c \le 0$, $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

$$\begin{cases} \partial_t^\alpha (u - a) = -Au \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u(\cdot, T) = b. \end{cases}$$

§5.1. Basic result

$$u(x,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha)(P_n a)(x), \quad a \in L^2(\Omega)$$

$\Longrightarrow$

$$b = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n T^\alpha) P_n a \quad \text{in } \Omega.$$

Then $P_n b = E_{\alpha,1}(-\lambda_n T^\alpha) P_n a$ for $n \in \mathbb{N}$.

Complete monotonicity of $E_{\alpha,1}(-\lambda_n t^\alpha)$ for $0 < \alpha \le 1$:

$$E_{\alpha,1}(-\lambda_n t^\alpha) \ge 0, \quad \frac{d}{dt}(E_{\alpha,1}(-\lambda_n t^\alpha)) \le 0, \quad t \ge 0.$$

Trivial for $\alpha = 1$.

$\Longrightarrow E_{\alpha,1}(-\lambda_n t^\alpha) > 0$ for $0 \le t \le T$.

Proof. $\exists\, t_0 > 0$ such that $E_{\alpha,1}(-\lambda_n t_0^{\alpha}) = 0$. Since $E_{\alpha,1}(-\lambda_n t^{\alpha})$ is monotone decreasing, we see $E_{\alpha,1}(-\lambda_n t^{\alpha}) = 0$ for $t \geq t_0$.
Analyticity implies $E_{\alpha,1}(-\lambda_n t^{\alpha}) = 0$ for all $t > 0$: contradiction. ■

Asymptotics:

$$E_{\alpha,1}(-\lambda_n T^\alpha) = \frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda_n T^\alpha} + O\left(\frac{1}{\lambda_n^2 T^{2\alpha}} \right).$$

$\Longrightarrow$

$$\frac{1}{E_{\alpha,1}(-\lambda_n T^\alpha)} = \lambda_n T^\alpha \left(\frac{1}{\Gamma(1-\alpha)} + O\left(\frac{1}{\lambda_n T^\alpha} \right) \right)^{-1} \sim \lambda_n T^\alpha$$

for large $T > 0$. Moreover $E_{\alpha,1}(-\lambda_n T^\alpha) > 0$. From

$$\frac{P_n b}{E_{\alpha,1}(-\lambda_n T^\alpha)} = P_n a$$

we have

$$\lambda_n^2 T^{2\alpha} ||P_n b||^2_{L^2(\Omega)} \sim ||P_n a||^2_{L^2(\Omega)}$$

that is, $||b||_{H^2(\Omega)\cap H_0^1(\Omega)} \sim ||a||_{L^2(\Omega)}$.

Thus we proved

Theorem 5.1 (Sakamoto-Yamamoto: 2011).

For each $b \in H^2(\Omega) \cap H_0^1(\Omega)$, *there exists a unique* $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ *such that*

$$\begin{cases} \partial_t^\alpha (u - a) = -Au, \quad u|_{\partial\Omega} = 0, \\ u(\cdot, T) = b. \end{cases}$$

Moreover $\|u(\cdot,0)\|_{L^2(\Omega)} \sim \|u(\cdot,T)\|_{H^2(\Omega)\cap H_0^1(\Omega)}$.

Remark.

- $0 < \alpha < 1$: memory effect and weak smoothing of ∂_t^α admit the backward well-posedness.
- Essentially different from $\alpha = 1$.

§5.2. Non self-adjoint A

$$-Lu := \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + \sum_{j=1}^{n} b_j(x)\partial_j u(x) + c(x)u$$

$$=: -Au + Bu$$

$$\begin{cases} \partial_t^\alpha (u-a) = -Lu \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u(T) = b. \end{cases}$$

Theorem 5.2.

For $b \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists unique solution a and u with same regularity as in Theorem 5.2.

Key to Proof

First Step.

$\partial_t^\alpha (u - a) = -Au + Bu$ in Ω and $u|_{\partial\Omega} = 0 \Longrightarrow$

$u_a(t) = S(t)a + \int_0^t K(t-s)Bu_a(s)ds$, where $S(t)$, $K(t)$ are constructed for $-A \Longrightarrow$

$b = S(T)a + \int_0^T K(T-s)Bu_a(s)ds \Longleftrightarrow$

$$a = S(T)^{-1}\left(b - \int_0^T K(T-s)Bu_a(s)ds\right)$$

$$= S(T)^{-1}b - S(T)^{-1}\int_0^T K(T-s)Bu_a(s)ds = S(T)^{-1}b - Ma$$

Here $Ma := S(T)^{-1}\int_0^T K(T-s)Bu_a(s)ds$

Unique solvability for a?

Second Step: compactness of $M : L^2(\Omega) \longrightarrow L^2(\Omega)$.

We can prove $\|Au(t)\| \le Ct^{-\alpha}\|a\|$ (Sakamoto-Yamamoto for $B = 0$)

Let $0 < \delta < \frac{1}{2}$.

$$\left\|A^{1+\delta}\int_0^T K(T-s)Bu_a(s)ds\right\| = \left\|\int_0^T A^{\frac{1}{2}+\delta}K(T-s)A^{\frac{1}{2}}Bu_a(s)ds\right\|$$

$$\le C\int_0^T (T-s)^{\alpha(\frac{1}{2}-\delta)-1}\|Au_a(s)\|ds \le C\int_0^T (T-s)^{\alpha(\frac{1}{2}-\delta)-1}s^{-\alpha}ds\|a\| \le C\|a\|.$$

$\Longrightarrow$

$$\left\|A^{\delta}S(T)^{-1}\int_0^T K(T-s)Bu_a(s)ds\right\| \le C\left\|A^{1+\delta}\int_0^T K(T-s)Bu_a(s)ds\right\| \le C\|a\|.$$

$\Longrightarrow M : L^2(\Omega) \longrightarrow H^{2\delta}(\Omega)$ is bounded

$M : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact.

Third Step:

Fredholm equation of second kind:

$$a = S(T)^{-1} b - Ma$$

For well-posedness, it suffices to verify that "$b = 0 \Longrightarrow a = 0$"

This is equivalent to Backward uniqueness:

$$\begin{cases} \partial_t^\alpha (u - a) = -Lu \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u(\cdot, T) = 0 \end{cases} \Longrightarrow \quad a = 0.$$

Fourth Step: Backward uniqueness.

P_m, $m \in \mathbb{N}$: eigenprojection for eigenvalue λ_m of L

Closed subspace spanned by all the generalized eigenfunctions is $L^2(\Omega)$ (Agmon)

$\Longrightarrow$ Therefore we see that $P_m a = 0$, $m \in \mathbb{N}$ imply $a = 0$.

Set

$$u_m(t) := P_m u(t), \quad a_m := P_m a$$

$\Longrightarrow$

$\partial_t^\alpha (u_m - a_m) = (-\lambda_m + D_m) u_m =: J u_m$ (Jordan form) where $D_m^\ell = 0$ with large ℓ.

$$\Longrightarrow \qquad u_m(t) = E_{\alpha,1}(Jt^\alpha) a_m = \sum_{k=0}^{\infty} \frac{(Jt^\alpha)^k}{\Gamma(\alpha k + 1)} a_m$$

Set $u_m = (u_m^1, ..., u_m^N)^T$ and $a_m = (a_m^1, ..., a_m^N)^T$ where ^{T}: transpose
We have a linear system:

$$u_m^1(t) = E_{\alpha,1}(-\lambda_m t^\alpha) a_m^1 + \cdots$$

$$u_m^2(t) = E_{\alpha,1}(-\lambda_m t^\alpha) a_m^2 + \cdots$$

$$\cdots\cdots$$

$$u_m^N(t) = E_{\alpha,1}(-\lambda_m t^\alpha) a_m^N$$

$$u(T) = 0 \Longrightarrow u_m(T) = 0$$

We have to prove that the linear system has only zero solution.

Use the backward substitution:
By $E_{\alpha,1}(-\lambda_m T^\alpha) \neq 0 \Longrightarrow a_m^N = 0$, then
$a_m^{N-1} = 0$, then .....
$a_m^1 = 0$.
$\Longrightarrow a_m = 0$ for $m \in \mathbb{N}$.

Thus $a = 0$!

§6. Fractional diffusion equations backward in time for $1 < \alpha < 2$

with Profs G. Floridia (Università Mediterranea di Reggio Calabria)

Recall $-Av(x) = \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j v(x)) + c(x)v(x)$ with $\mathcal{D}(A) := H^2(\Omega) \cap H_0^1(\Omega)$.

Direct problem:

$$\begin{cases} \partial_t^\alpha (u(x,t) - a(x) - b(x)t) = -Au(x,t), \quad x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = 0. \end{cases}$$

$0 < \lambda_1 < \lambda_2 < \cdots \longrightarrow \infty$: set of eigenvalues

Let P_n: orthogonal projection into Ker $(A - \lambda_n)$.

Proposition (direct problem).

Let $a, b \in L^2(\Omega)$. Then there exists a unique solution $u_{a,b}$ such that

$$\begin{cases} u_{a,b} \in C([0,T]; L^2(\Omega)) \cap C((0,T; H^2(\Omega) \cap H_0^1(\Omega)), \\ \lim_{t\to 0} \|u(\cdot,t) - a\|_{L^2(\Omega)} = \lim_{t\to 0} \|\partial_t u(\cdot,t) - b\|_{H^{-2}(\Omega)} = 0 \end{cases}$$

and

$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} \{E_{\alpha,1}(-\lambda_n t^{\alpha}) P_n a(x) + t E_{\alpha,2}(-\lambda_n t^{\alpha}) P_n b(x)\} \\ \qquad \text{in } C([0,T]; L^2(\Omega)), \\ \partial_t u(x,t) = \sum_{n=1}^{\infty} \{-\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) P_n a(x) + E_{\alpha,1}(-\lambda_n t^{\alpha}) P_n b(x)\} \\ \qquad \text{in } C([0,T]; L^2(\Omega)). \end{cases}$$

Backward problem:

Let $T > 0$ and a_T, b_T. Find $u = u(x,t)$, and $a, b \in L^2(\Omega)$ such that

$$\begin{cases} \partial_t^\alpha (u - a - bt) = -Au, \quad x \in \Omega,\ t > 0, \\ u(\cdot, T) = a_T, \quad \partial_t u(\cdot, T) = b_T, \qquad x \in \Omega \end{cases}$$

Theorem 6.1 (generic well-posedness).
There exists a finite set $\{\eta_1, ..., \eta_N\} \subset (0, \infty)$ *satisfying:*
(i) If

$$T \notin \bigcup_{n=1}^{\infty} \left\{ \left(\frac{\eta_1}{\lambda_n}\right)^{\frac{1}{\alpha}}, \cdots, \left(\frac{\eta_N}{\lambda_n}\right)^{\frac{1}{\alpha}} \right\},$$

then for any $a_T, b_T \in H^2(\Omega) \cap H_0^1(\Omega)$, *there exist* $a, b \in L^2(\Omega)$ *such that the solution* $u_{a,b}$ *satisfies* $u_{a,b}(\cdot, T) = a_T$ *and* $\partial_t u_{a,b}(\cdot, T) = b_T$. *Moreover*

$$\|a_T\|_{H^2(\Omega)} + \|b_T\|_{H^2(\Omega)} \sim \|a\|_{L^2(\Omega)} + \|b\|_{L^2(\Omega)}$$

for all $a_T, b_T \in L^2(\Omega)$
(ii) No uniqueness for backward problem for

$$T \in \bigcup_{n=1}^{\infty} \left\{ \left(\frac{\eta_1}{\lambda_n}\right)^{\frac{1}{\alpha}}, \cdots, \left(\frac{\eta_N}{\lambda_n}\right)^{\frac{1}{\alpha}} \right\}.$$

Properties of the exceptional set of

$$\bigcup_{n=1}^{\infty}\left\{\left(\frac{\eta_1}{\lambda_n}\right)^{\frac{1}{\alpha}},\dots,\left(\frac{\eta_N}{\lambda_n}\right)^{\frac{1}{\alpha}}\right\}$$

- $\mathbf{0}$ is a unique accumulation point
- included in

$$\left[0,\left(\frac{\eta_N}{\lambda_1}\right)^{\frac{1}{\alpha}}\right]$$

$\Longrightarrow$ backward problem is well-posed for $T > \left(\frac{\eta_N}{\lambda_1}\right)^{\frac{1}{\alpha}}$

Summing-up for Backward problem in time.

- **$0 < \alpha < 1$:** well-posed for any $T > 0$.
- **$\alpha = 1$:** severely ill-posed but uniqueness and conditional stability for any $T > 0$.
- **$1 < \alpha < 2$:** well-posed for $T > 0$ not belonging to a countably infinite set. Even non-uniqueness occurs for such exceptional values of T.
- **$\alpha = 2$:** Well-posed. Also conservation quantity such as energy, which is impossible for $\alpha \neq 2$.

Sketch of Proof.
$u(\cdot,T) = a_T$ and $\partial_t u(\cdot,T) = b_T \Longleftrightarrow$

$$\begin{cases} E_{\alpha,1}(-\lambda_n T^\alpha)P_n a + TE_{\alpha,2}(-\lambda_n T^\alpha)P_n b = P_n a_T, \\ -\lambda_n T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n T^\alpha)P_n a + E_{\alpha,1}(-\lambda_n T^\alpha)P_n b = P_n b_T \end{cases}$$

$\Longrightarrow$ linear system w.r.t. $P_n a$ and $P_n b$.

Determinant of coefficient matrix $:= \psi(\lambda_n T^\alpha)$:

$$\psi(\eta) := E_{\alpha,1}(-\eta)^2 + \eta E_{\alpha,2}(-\eta)E_{\alpha,\alpha}(-\eta), \quad \eta > 0$$

Well-posed of backward problem $\Longleftrightarrow \psi(\lambda_n T^\alpha) \neq 0$
Asymptotics of $\psi \Longrightarrow \psi(\infty) < 0$.
Moreover by $\psi(0) = 1 > 0$, intermediate value theorem implies $\exists \eta_0$ satisfying $\psi(\eta_0) = 0$
ψ: analytic in $\eta > 0 \Longrightarrow$ finite zeros: $\eta_1, \ldots, \eta_N \in (0,\infty)$

$\eta(\eta_k) = 0 \Longleftrightarrow \lambda_n T^\alpha = \eta_k$, i.e., $T = \left(\frac{\eta_k}{\lambda_n}\right)^{\frac{1}{\alpha}}$. ■

§7. Inverse source problem without boundary conditions: Formulation and main results

$$\partial_t^\alpha y(x,t) + Ay(x,t) = \mu(t)f(x), \quad y(x,0) = 0, \quad x \in \Omega,\ 0 < t < T.$$

Here $\Omega \subset \mathbb{R}^d$, $-A$: time independent uniform elliptic operator:
$-Av(x) = \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j v) + \sum_{k=1}^{d} b_k(x)\partial_k v + c(x)v,\ 0 < \alpha < 1.$

Let $\omega \subset \Omega$ be subdomain and μ given.
Uniqueness in inverse source problem. $y|_{\omega\times(0,T)} = 0 \Longrightarrow f = 0$ *in* Ω?

References: many, but assuming boundary conditions.

We should NOT assume boundary conditions.
$\Longrightarrow$ Unique continuation: Without boundary conditions.

$$\begin{cases} \partial_t^\alpha (u-a) + Au = 0 \quad \text{in } \Omega \times (0,T), \\ u|_{\omega\times(0,T)} = 0 \quad \Longrightarrow \quad a = 0 \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{in } \Omega \times (0,T). \end{cases} \tag{$*$}$$

Correspondingly we should prove

$$\begin{cases} \partial_t^\alpha y + Ay = \mu(t) f(x) \quad \text{in } \Omega \times (0,T), \\ y|_{\omega\times(0,T)} = 0, \quad y(\cdot,0) = 0 \quad \text{in } \Omega \end{cases} \tag{$**$}$$

$\Longrightarrow f = 0$ in Ω!

Key: Duhamel principle connecting (*) and (**)

Theorem 7.1.

Consider $\partial_t^\alpha y + Ay = \mu(t)f(x)$ *in* $\Omega \times (0,T)$.

We assume (i) or (ii):

(i) $\mu \in C^1[0,T]$, $\mu(0) \neq 0$ *and* $y = y(x,t) \in H_\alpha(0,T;H^2(\Omega))$

(ii) $\mu(t) = t^{\beta-1}K(t)$ *where* $\frac{dK}{dt} \in L^\infty(0,T)$, $K(0) \neq 0$, $\beta \leq 1-\alpha$ *and* $y = y(x,t) \in H_\alpha(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$.

Then $y|_{\omega\times(0,T)} = 0$ *implies* $f = 0$ *in* Ω.

Remark.

Case (i) $\mu \in C^1$: Mildly activated source

Case (ii) $\mu(t) = t^{\beta-1}K(t)$ with $\beta < 1$: Explosively activated source

- No boundary condition
- $1 \le \alpha < 2$: same conclusion
- Same with Neumann data $\partial_\nu y|_{\gamma\times(0,T)}$ with subboundary γ
- Regularity assumption on $y \in H_\alpha(0,T;H^2(\Omega))$ is essential for mildly activated source case. Usually $y \in H_\alpha(0,T;L^2(\Omega)) \cap L^2(0,T : H^2(\Omega) \cap H_0^1(\Omega))$ can be expected and in Theorem 7.1 we need more regularity.

Example.

$$\partial_t^\alpha y - \Delta y = \mu(t) f(x), \quad y|_{\partial\Omega} = 0.$$

If $f \in H^2(\Omega) \cap H_0^1(\Omega)$, then $y \in H_\alpha(0,T;H^2(\Omega))$.

Unique continuation: key

Let $u \in L^2(0,T;H^2(\Omega))$ satisfy $u - f \in H_\alpha(0,T;L^2(\Omega))$ and $\partial_t^\alpha(u-f) + Au = 0$.
Then $u|_{\omega\times(0,T)} = 0$ implies $u \equiv 0$ and $f \equiv 0$.

References on time-fractional diffusion equations:

- Lin-Nakamura 2022: general dimensions, multi-term fractional equation, $0 < \alpha < 2$, $\alpha \neq 1$.
- Li-Liu-Yamamoto 2022:
  1-dim. for weaker solution $u - f \in H_\alpha(0,T;H^{-1}(0,1))$ and $u \in L^2(0,T;H^1(0,1))$.

Sketch of Proof.

Key idea:

Through Duhamel principle, define solution $u(x,t)$ to $\partial_t^\alpha(u-f)+Au=0$ for y!

First Step.

Solve $u(x,t)$ in

$$J^{1-\alpha}y(\cdot,t)=\int_0^t \mu(t-s)u(x,s)ds=:(\mu*u)(x,t).$$

Solvability of convolution equation with kernel μ is essential.

Case (i): $\mu \in C^1[0,T]$ with $\mu(0) \neq 0$

Recall: $J^{1-\alpha}y = \mu * u$.

$\mu(0) \neq 0 \Longrightarrow u \in L^2(0,T;H^2(\Omega))$ exists uniquely by Volterra equation of second kind for $y \in H_\alpha(0,T;H^2(\Omega))$:

$$(J^{1-\alpha}y)'(\cdot,t) = \mu(0)u(x,t) + \int_0^t \mu'(t-s)u(x,s)ds$$

Here we used

Lemma 7.1. *(i)* $J^\alpha : H_p(0,T) \Longrightarrow H_{p+\beta}(0,T)$ *is isomorphism with* $p \geq 0$.

(ii) $z \in H_\alpha(0,T)$ *implies* $J^{1-\alpha}z \in H_1(0,T)$.

Second Step.

Recall $J^{1-\alpha}y = \mu * u$

$\Longrightarrow D_t^{1-\alpha}J^{1-\alpha}y(t) = D_t^{1-\alpha}(\mu * u)(t) = (D_t^{1-\alpha}\mu * u)(x,t)$

$\Longrightarrow y(t) = (\rho * u)(x,t)$ where $\rho(t) := D_t^{1-\alpha}\mu(t)$ (Not write $\mu = J^{1-\alpha}\rho$):

Here we used

Lemma 7.2. *(i)* $D_t^{1-\alpha}J^{1-\alpha}g = g$ *for* $g \in L^1(0,T)$.

(ii) $D_t^{1-\alpha}(\mu * g) = D_t^{1-\alpha}\mu * g$ *for* $\mu \in C^1[0,T]$, $g \in L^2(0,T)$.

$u(x,t)$ is expected to satisfy $\partial_t^\alpha(u - f) + Au = 0$.

Note: $\partial_t^\alpha y + Ay = \mu(t) f(x) \Longleftrightarrow y + J^\alpha Ay = (J^\alpha \mu) f$.

Substitute $y(t) = (\rho * u)(x, t)$ and $\rho(t) := D_t^{1-\alpha} \mu(t)$.

$\Longrightarrow$

$$(\rho * u)(x, t) + J^\alpha (\rho * Au)(x, t) = (J^\alpha \mu)(t) f(x)$$

$\Longrightarrow$

$$(\rho * u)(x, t) + (\rho * J^\alpha Au)(x, t) = (J^\alpha \mu)(t) f(x).$$

Here we used

Lemma 7.3. $J^\alpha (\rho * g) = (\rho * J^\alpha g)$ *for* $\rho, g \in L^1(0, T)$.

We obtained: $(\rho * u)(x,t) + (\rho * J^{\alpha} Au) = (J^{\alpha}\mu)(t) f(x)$.

We prove $\rho = D_t^{1-\alpha}\mu \Longrightarrow J^{\alpha}\mu(t) = J^1\rho(t) = \int_0^t \rho(t-s)ds$
$\Longrightarrow (\rho * (u + J^{\alpha}Au - f))(x,t) = 0,\ 0 < t < T$.

$\rho \not\equiv 0$ and Titchmarsh theorem yields $(u-f) + J^{\alpha}Au = 0$ in $(0, \exists t_*) \Longrightarrow$
$\partial_t^{\alpha}(u-f) + Au = 0$ in $\Omega \times (0, t_*)$.

Recall $0 = J^{1-\alpha}y(t) = \int_0^t \mu(t-s)u(x,s)ds$ in $\omega \times (0,T)$. Hence $u|_{\omega\times(0,T)} = 0$.

Unique continuation $\Longrightarrow u \equiv 0$ and $f \equiv 0$.

Concluding remarks.

(I) Essence is the solution $v \in H_\alpha(0,T)$ satisfying

$$z(t) = \int_0^t \mu(t-s)v(s)ds.$$

We consider two cases for μ.

(II) Let $\mu \in C^\infty[0,T]$, $\mu(0) = 0$ but $\mu'(0) \neq 0$ in $\partial_t^\alpha y + Ay = \mu(t)f$ in $\Omega \times (0,T)$.

$J^{1-\alpha}y(\cdot,t) = \int_0^t \mu(t-s)u(\cdot,s)ds$

$\Longrightarrow \partial_t^2 J^{1-\alpha}y(\cdot,t) = \mu'(0)u(\cdot,t) + \int_0^t \mu''(t-s)u(\cdot,s)ds$

Hence smoother y implies $\exists u \in H_\alpha(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$.

However, we cannot assume so strong regularity to y for time fractional equations.

(III) Same for $1 \le \alpha \le 2$. Similar for $\sum_{k=1}^{N} q_j \partial_t^{\alpha_j} y + Ay = \mu(t) f(x)$ where $q_j \in \mathbb{R}$ and $0 < \alpha_j < 2$

(IV) Theorem 7.2 (uniqueness for inverse heat source problem without boundary condition)
Assume $\mu \in C^1[0,T]$ and $\mu(0) \ne 0$ in $\partial_t y + Ay = \mu(t) f(x)$ and $y(\cdot, 0) = 0$ in Ω.
Then $y|_{\omega\times(0,T)} = 0$ implies $f = 0$ in Ω.

Even this uniqueness is novel.

General form:

$$\partial_t u + Au = R(x,t) f(x), \quad u(\cdot, t_0) = 0$$

without boundary conditions.

$R(x,t)$ depends on x, and assume $R(x, t_0) \ne 0$ for $x \in \overline{\Omega}$.

- Case $t_0 = 0$: hard open problem so far!
- Case $0 < t_0 < T$:
  Bukhgeim-Klibanov 1981, Imanuvilov and Yamamoto 1998, etc. by Carleman estimate
- Case $t_0 = T$: it was open but Imanuvilov and Yamamoto 2022 solved! by Carleman estimate

(V) $\alpha = 2$: we can directly discuss by hyperbolic Carleman estimate with $\mu(x,t)$: even extension of $y(\cdot,t)$ to $(-T,0)$ + Carleman estimate

(VI) Inverse problems for time fractional equations are:

- different from $\alpha = 1, 2$ if issue is temporal.
- similar to $\alpha = 1, 2$ if issue is spatial such as the current inverse problem

§8. Inverse source problem by data after incident

Motivation: data after incident

$$\begin{cases} \partial_t^\alpha u = \Delta u(x,t) + \mu(t) f(x), & x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = 0, \quad t > 0, & \text{with zero initial values,} \end{cases}$$

where $0 < \alpha < 2$ and also $\alpha = 1$ or 2.
$\mu(t) f(x)$: source of contaminants

Example: explosion of power plant $\Longrightarrow$ wide fatal diffusion of Cs-137
$\mu(t)$ is activated short period: $\mu(t) = 0$ for $t > \exists t_0$.

Usually observation can start only after the explosion,
not at $t = 0$.

Inverse source problem:

Determine $\mu(t)$ *or* $f(x)$ *by data* $\partial_\nu u$ *on* $\Gamma \times (t_0, T_0)$.

Here $\Gamma \subset \partial\Omega$: arbitrarlily chosen subboundary, and $0 < t_0 < T_0$

Many references for case $t_0 = 0$ (e.g., Yamamoto (1992), etc.):

Uniqueness for $t_0 = 0$

- $\alpha = 1$: heat equation: arbitrary $\Gamma \subset \partial\Omega$, $T_0 > 0$
- $\alpha = 2$: wave equation: large $\Gamma \subset \partial\Omega$, $T_0 > 0$

Main result

$\nu = (\nu_1, ..., \nu_d)$: unit outward normal vector to $\partial\Omega$, $d = 1, 2, 3$ (for simplicity),

Let $(-Av)(x) := \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j v) + c(x)v$: elliptic, where $a_{ij} = a_{ji}$, c are smooth, $c \le 0$,

and $\partial_\nu w := \sum_{i,j=1}^{d} a_{ij}(x)(\partial_j v)(x)\nu_i(x)$.

$$
\begin{cases}
\partial_t^\alpha u(x,t) = -Au + \mu(t)f(x), \quad x \in \Omega,\ t > 0, \quad u(x,t) = 0, x \in \partial\Omega,\ t > 0, \\
u(x,0) = 0, \quad x \in \Omega \quad \text{for } 0 < \alpha < 1, \\
u(x,0) = \partial_t u(x,0) = 0, \quad x \in \Omega \quad \text{for } 1 < \alpha < 2.
\end{cases}
$$

Setting:

$$
\begin{cases}
\Gamma \subset \partial\Omega\text{: subboundary, } 0 < t_0 < T_0\text{: arbitrarily chosen,} \\
\mu \in L^2_{loc}(0,\infty), \quad \mu(t) = 0 \quad \text{for } t > t_0,, \quad f \in H^2(\Omega) \cap H^1_0(\Omega),
\end{cases}
$$

Theorem 8.1.
$\partial_\nu u = 0$ *on* $\Gamma \times (t_0, T_0) \Longrightarrow f \equiv 0$ *in* Ω *or* $\mu \equiv 0$ *in* $(0, t_0)$.

References on fractional equation with data after incident:

- Kian-Liu-Yamamoto (2022) proved $u|_{\omega\times(t_0,T_0)} \Longrightarrow f$: uniqueness under extra assumption $f = 0$ in subdomain ω.
- Janno-Kian (2022): similar uniqueness by different method

Theorem 8.1 means:

Data for time-fractional diffusion-wave equations keep enough information of unknown sources after incident.

Very different from the heat equation ($\alpha = 1$) and the wave equation ($\alpha = 2$).

Uniqueness for inverse source problems for $\alpha = 1, 2$ by data after incident by Cheng, Lu and Yamamoto (2019).

(I) heat equation

$$\begin{cases} \partial_t u = \Delta u + \mu(t)f(x), \quad x \in \Omega,\ t > 0, \\ u(x,0) = 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0. \end{cases}$$

Let $\mu \in H^1_{loc}(0,\infty)$, $\mu(t) = 0$ for $t > t_0$, $f \in C_0^\infty(\Omega)$ (for simplicity).
$\{\lambda_k\}_{k\in\mathbb{N}}$; set of eigenvalues $A := -\Delta$ with $v|_{\partial\Omega} = 0$.
P_k: orthogonal projection of $L^2(\Omega)$ into Ker $(A - \lambda_k)$. Set $\Lambda := \{k \in \mathbb{N};\ P_k f = 0$ in $\Omega\}$.

Theorem 8.2 (uniqueness for $\mu(t)$ in heat equation).
Let $\Gamma \subset \partial\Omega$, $0 < t_0 < T_0$*: arbitrarily given. Then*

$$\sum_{k\in\mathbb{N}\setminus\Lambda} \frac{1}{\lambda_k} = \infty$$

if and only if $\partial_\nu u = 0$ *on* $\Gamma \times (t_0, T_0)$ *implies* $\mu = 0$ *in* $(0, t_0)$.

Corollary. *No uniqueness in case* $d = 1$ *(1-D).*

Because: $\lambda_k \sim k^2 \Longrightarrow \sum_{k\in\mathbb{N}} \frac{1}{\lambda_k} < \infty$.

"Boundary data in one dimensional case after incident have no spatial component and so enough information is not preserved after incident in e.g., $u_x(0,t)$, $t > t_0$."

Remark on variant of Theorem 8.2. Let $T > t_0$ be arbitrarily fixed.

$$\begin{cases} \partial_t u = \Delta u + \mu(t) f(x), & x \in \Omega,\ t > 0, \\ u(x,0) = 0, \quad x \in \Omega, & u|_{\partial\Omega} = 0. \end{cases}$$

$u(x,T)$, $x \in \Omega$ determines $\mu(t)$, $0 < t < t_0$ uniquely under the same assumption of Theorem 8.2.

Proof. $u(\cdot, T) = 0 \Longrightarrow u = 0$ in $\Omega \times (T, \infty)$.

Backward uniqueness $\Longrightarrow u = 0$ in $\Omega \times (t_0, T)$.

Theorem 8.2 implies uniqueness.

Theorem 8.3 (uniqueness for $f(x)$ in heat equation).

$\Gamma \subset \partial\Omega$, $0 < t_0 < T_0$: arbitrarily given. Then

$$\int_0^{t_0} e^{\lambda_k t} \mu(t) dt \ne 0, \quad \forall k \in \mathbb{N}$$

if and only if $\partial_\nu u = 0$ on $\Gamma \times (t_0, T_0)$ implies $f = 0$ in Ω.

Remark. Conditional logarithmic stability is possible if $\mu \ge 0, \not\equiv 0$.

(II) wave equation

$$\begin{cases} \partial_t^2 u = \Delta u + \mu(t) f(x), \quad x \in \Omega,\ t > 0, \\ u(x,0) = \partial_t u(x,0) = 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0. \end{cases}$$

Let $\mu \in H^1_{loc}(0,\infty)$, $\mu(t) = 0$ for $t > t_0$, $f \in C_0^\infty(\Omega)$ (for simplicity).

Theorem 8.4 (uniqueness for $\mu(t)$ in wave equation). *Assume*

$$\text{(i)} \quad \lim_{k\to\infty, k\in\mathbb{N}\setminus\Lambda} \frac{k}{\sqrt{\lambda_k}} > 0.$$

$$\text{(ii)} \quad \exists x_0 \in \mathbb{R}^d \text{ such that } \Gamma \supset \{x \in \partial\Omega;\ (x - x_0)\cdot\nu \ge 0\}$$

$$\text{(iii)} \quad T_0 - t_0 > 2\sup_{x\in\Omega} |x - x_0|, \quad t_0 < \pi \lim_{k\to\infty, k\in\mathbb{N}\setminus\Lambda} \frac{k}{\sqrt{\lambda_k}}.$$

Then $\partial_\nu u = 0$ on $\Gamma \times (t_0, T_0)$ implies $\mu = 0$ in $(0, t_0)$.

Theorem 8.5 (uniqueness for $f(x)$ in wave equation).
$\Gamma \subset \partial\Omega$, $T_0 - t_0 > 0$ sufficiently large, $t_0 > 0$ sufficiently small (as in Theorem 4).
Assume that $\mu(0) \neq 0$ and

$$\left|\int_0^{t_0} \mu(s) \sin \sqrt{\lambda_k}(t_0 - s)ds\right| + \left|\int_0^{t_0} \mu(s) \cos \sqrt{\lambda_k}(t_0 - s)ds\right| \neq 0, \forall k \in \mathbb{N}.$$

Then $\partial_\nu u = 0$ on $\Gamma \times (t_0, T_0)$ implies $f = 0$ in Ω.

Moreover we have both-sided global Lipschitz stability

Comparison of uniqueness results

- $\alpha = 1$ or $\alpha = 2$: complicated
- $\alpha \notin \mathbb{N}$: always sharp uniqueness

Why?

- Phenomenologically $\Longrightarrow$ ∂_t^α with $\alpha \notin \mathbb{N}$ has memory effects.
- Mathematically $\Longrightarrow$

Differences of time factors of solution formulae:

$\alpha = 1$: $e^{-\lambda_k t}$

$\alpha = 2$: $\sin \sqrt{\lambda_k} t$

$\alpha \notin \mathbb{N}$: Mittag-Leffler functions

Key for proofs of uniqueness for heat and wave inverse source problems.

- $\alpha = 2$: $\{\sin \sqrt{\lambda_k} t\}_{k\in\mathbb{N}}$ is linearly independent in $L^2(0, t_0)$?
  and observability inequality.
  We rely on non-harmonic Fourier series.
- $\alpha = 1$: $\{e^{-\lambda_k t}\}_{k\in\mathbb{N}}$ is linearly independent in $C[0, t_0]$?
  $\Longleftarrow$ Müntz theoerm (Weierstrass polynomial approximation theorem)

In particular, 1-D case implies non-uniqueness.

Key Proposition for the proof of main result

Let $E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\alpha)}$, P_k: orthogonal projection to ker $(A - \lambda_k)$. Then

$$u(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^{\alpha}) \mu(s) ds \right) (P_n f)(x)$$

in $L^2(0,T; H^{2-\varepsilon}(\Omega))$ with small $\varepsilon > 0$.

Choose T_1 such that $t_0 < T_1 < T_0$. Then $\mu(t) = 0$ for $t > t_0$ implies

$$u(x,t) = \sum_{n=1}^{\infty} \psi_n(t)(P_n f)(x), \quad t \ge T_1.$$

Here

$$\psi_n(t) := \int_0^{t_0} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^{\alpha}) \mu(s) ds, \quad t \ge T_1.$$

Recall

$$\psi_n(t) := \int_0^{t_0} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^{\alpha}) \mu(s) ds, \quad t \ge T_1.$$

$\{\lambda_n\}_{n\in\mathbb{N}}$: set of eigenvalues of $-A$ with $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

Key Proposition: linear independence of $\{\psi_n\}_{n\in\mathbb{N}}$ in (T_1, ∞).

Assume $\mu \not\equiv 0$ *in* $(0, t_0)$ *and* $\sum_{n=1}^{\infty} \left| \frac{a_n}{\lambda_n} \right| < \infty$. *Then*

$$\sum_{n=1}^{\infty} a_n \psi_n(t) = 0 \quad \text{for } t > T_1$$

implies $a_n = 0$ *for all* $n \in \mathbb{N}$.

How Theorem 8.1 is derived from Key Proposition?

Neglecting the convergence issues (justified), we sketch.

Let $\mu \not\equiv 0$ in $(0, t_0)$.
$\psi_n(t)$: analytic for $t > T_1 \Longrightarrow$
$\partial_\nu u = 0$ on $\Gamma \times (t_0, T_0)$ implies

$$\partial_\nu u(x,t) = \sum_{n=1}^{\infty} \psi_n(t) \partial_\nu (P_n f)(x) = 0 \quad \text{for } (x,t) \in \Gamma \times (T_1, \infty).$$

Key Proposition $\Longrightarrow \partial_\nu(P_n f) = 0$ on Γ for all $n \in \mathbb{N}$.
Note $(A - \lambda_n)P_n f = 0$ in Ω and $P_n f = 0$ on $\partial\Omega$.

Unique continuation for elliptic operator $A - \lambda_n \Longrightarrow P_n f = 0$ in Ω for all $n \in \mathbb{N}$.

Therefore $f = 0$ in Ω. ■

Sketch of proof of Key Proposition

Set $\psi_n(t) := \int_0^{t_0} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha)\mu(s)ds$ and $A_k := \sum_{n=1}^\infty \frac{a_n}{\lambda_n^k}$.

Substitute asymptotic expansion for $K \in \mathbb{N}$: as $t \to \infty$ we have

$$\eta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) = \sum_{k=2}^K \frac{(-1)^{k+1}}{\Gamma(\alpha-\alpha k)} \frac{1}{\lambda_n^k t^{\alpha k-\alpha+1}} + O\left(\frac{1}{\lambda_n^{K+1} t^{\alpha(K+1)-\alpha+1}}\right).$$

Here no terms if $\alpha - \alpha k = -1, -2, ...$ Assume $\alpha \notin \mathbb{Q}$ for simplicity. Then

$$\left|\sum_{n=1}^\infty a_n \psi_n(t) - \sum_{n=1}^\infty a_n \sum_{k=2}^K \frac{(-1)^{k+1}}{\Gamma(\alpha-\alpha k)} \frac{1}{\lambda_n^k} \int_0^{t_0} \frac{\mu(s)}{(t-s)^{\alpha k-\alpha+1}} ds\right|$$

$$\le \sum_{n=1}^\infty \frac{|a_n|}{\lambda_n^{K+1}} \int_0^{t_0} \frac{C_K}{(t-s)^{\alpha(K+1)-\alpha+1}} |\mu(s)|ds, \quad t > T_1.$$

Hence, for $t > T_1$, $\sum_{n=1}^{\infty} a_n \psi_n(t) = 0 \Longrightarrow$

$$\left| \sum_{k=2}^{K} \frac{(-1)^{k+1}}{\Gamma(\alpha - \alpha k)} \left(\int_0^{t_0} \frac{\mu(s)}{(t-s)^{\alpha k - \alpha + 1}} ds \right) A_k \right| \leq C \int_0^{t_0} \frac{|\mu(s)|}{(t-s)^{\alpha(K+1) - \alpha + 1}} ds$$

Substitute Binomial expansion:

$$\int_0^{t_0} \frac{\mu(s)}{(t-s)^\sigma} ds = \sum_{m=0}^{M} \begin{pmatrix} -\sigma \\ m \end{pmatrix} \frac{\mu_m}{t^{\sigma+m}} + \frac{C(M,\sigma)}{t^{\sigma+M+1}}$$

for $\sigma = \alpha+1, 2\alpha+1, 3\alpha+1...$ and $M \in \mathbb{N}$. Here we set $\mu_m := \int_0^{t_0} (-s)^m \mu(s) ds$.

Then $\mu \not\equiv 0$ in $(0, t_0) \Longrightarrow \exists m_1 \in \mathbb{N}$ such that $\mu_{m_1} = \int_0^{t_0} (-s)^{m_1} \mu(s) ds \neq 0$.

$$\left| \sum_{k=2}^{K} \frac{(-1)^{k+1}}{\Gamma(\alpha - \alpha k)} \left\{ \sum_{m=m_1}^{M} \binom{-\alpha k + \alpha - 1}{m} \frac{\mu_m}{t^{\alpha k - \alpha + 1 + m}} + \frac{C(M,K)}{t^{\alpha k - \alpha + M + 2}} \right\} A_k \right|$$

$$\leq \frac{\widetilde{C}(M,K)}{(t - t_0)^{\alpha(K+1) - \alpha + 1}} \quad \text{as } t \to \infty.$$

Comparison of coefficients of terms $\frac{1}{t^{\alpha k - \alpha + 1 + m_1}}$

$\Longrightarrow A_k := \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^k} = 0$ for all $k \geq 2, \in \mathbb{N}$.

Lemma 8.1 (again!)

$\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^k} = 0$ *for all* $k \in \mathbb{N}, \geq 2$ *implies* $a_n = 0$ *for all* $n \in \mathbb{N}$.

The proof of Key Proposition is complete. ■

§9. Inverse coefficient problems

Let $0 < \alpha < 1$, $\Omega \subset \mathbb{R}^d$: bounded,
$\Gamma \subset \Omega$: arbitrarily fixed subboundary.

$$u_p(x,t) \begin{cases} \partial_t^\alpha u = \Delta u + p(x)u(x,t), \quad x \in \Omega \subset \mathbb{R}^d, \\ \partial_\nu u|_{\partial\Omega} = 0, \quad u(x,0) = a(x), \quad x \in \Omega \end{cases}$$

Inverse coefficient problem.
$\partial_\nu u|_{\Gamma\times(0,T)} \Longrightarrow p(x)$, $x \in \Omega$.

Similar result with data $\partial_\nu u|_{\Gamma\times(0,T)}$ with $\Gamma \subset \partial\Omega$.

Limited methodologies:

I. Spectral approach for 1D case:
Cheng-Nakagawa-Y.-Yamazaki (2009), Jin and Rundell (2012),
Li-Zhang-Jia-Yamamoto (2013), Ren, Huang and Y. (2021), Jing and Y. (2022),etc.

Theorem 9.1 (Rundell and Y. 2021) *For* $p \le 0, \in C[0,1]$, *let*

$$u_{p,\alpha} \quad \begin{cases} \partial_t^\alpha u = \partial_x^2 u + p(x)u(x,t), \quad 0 < x < 1,\ t > 0, \\ \partial_x u(0,t) = 0, \quad \partial_x u(1,t) = g(t) \not\equiv 0. \end{cases}$$

Let $g \not\equiv 0, \in H_{\max\{\alpha,\beta\}}(0,T)$. *Then* $u_{p,\alpha}(1,t) = u_{q,\beta}(1,t),\ 0 < t < T$
$\Longrightarrow \alpha = \beta,\ p(x) = q(x),\ 0 < x < 1.$

Remark: Scenario for spectral approach.

This can work in 1D case.

Data of u *= [eigenfunction expansion]* $\Longrightarrow$ *Spectral data = [Gel'fand-Levitan theory]* $\Longrightarrow$ *Uniqueness*

Step 1:

(1) 1D case: well by $\lambda_n = c_0 n + O(1/n)$

(2) General dimensions: bottleneck

because only $\lambda_n = c_0 n^{2/d} + o(n^{2/d})$.

Step 2:

Gel'fand-Levitan theory: spectral data determine $p(x)$ uniquely (e.g., Nachman-Sylvester-Uhlmann)

II. Integral transform to hyperbolic equation:
Miller-Yamamoto (2013)

III. Dirichlet-to-Neumann map:
Imanuvilov-Li-Yamamoto (2016), Kian-Oksanen-Soccorsi-Yamamoto (2018), etc.

IV. special orders. $\partial_t^{1/2} - \partial_x^2$: 1D
Yamamoto-Zhang (2012) $\Longrightarrow$
$(\partial_t^{1/2} + \partial_x^2)(\partial_t^{1/2} - \partial_x^2) = \partial_t - \partial_x^4$: PDE of natural-number but higher order

Available methodologies

- No comprehensive methods for inverse coefficient problems for fractional differential equations
- Current methods: Reduction to
  1. elliptic inverse problem $\Longrightarrow$ spectral approach
  2. hyperbolic inverse problem $\Longrightarrow$ the method by Carleman estimates

Inverse coefficient problems via hyperbolic equations

Theorem 9.2 (Miller-Yamamoto: 2013).

Let $a > 0$ *on* $\overline{\Omega}$, $p \le 0$ *be smooth, and let the solution* $u_p(x,t)$ *be smooth:*

$$\partial_t^\alpha (u - a) = (\Delta + p(x))u, \quad \partial_\nu u|_{\partial\Omega} = 0.$$

If $u_p = u_q$ *on* $\partial\Omega \times (0, t_0)$ *with some* $t_0 > 0$, *then* $p = q$ *in* Ω.

Integral transform with Wright function (*Bazhlekova subordination formula*) can change time-fractional inverse problem to hyperbolic inverse problem.
cf. Reznitskaya transform: hyperbolic IP $\Longrightarrow$ parabolic IP

However, here we explain more naively.
Let

$$u_p : \partial_t^\alpha (u - a) = (\Delta + p(x))u, \quad \partial_\nu u|_{\partial\Omega} = 0$$

$$w_p : \begin{cases} \partial_t^2 w = (\Delta + p(x))w, \quad \partial_\nu w|_{\partial\Omega} = 0, \\ w(x,0) = 0, \quad \partial_t w(x,0) = a(x) \quad \text{in } \Omega. \end{cases}$$

By Laplace transform, we will switch u_p to w_p (using idea in Jiang, Li, Liu-Y.: 2017). We explain formally but can be justified for e.g., convergence, etc.

$(Lu)(x,s) := \widehat{u}(x,s) := \int_0^\infty e^{-pt} u(x,t)dt$.

Then $\partial_t^\alpha (u-a) = (\Delta + p(x))u_p \Longrightarrow$

$s^\alpha \widehat{u_p}(x,s) - s^{\alpha-1}a = (\Delta + p(x))\widehat{u_p}$, that is,

$s^\alpha (s^{1-\alpha}\widehat{u_p}) - a = (\Delta + p(x))(s^{1-\alpha}\widehat{u_p})$.

Set $\widetilde{v_p}(x,s) := s^{1-\alpha}\widehat{u_p}(x,s)$. Then $s^\alpha \widetilde{v_p} - a = (\Delta + p(x))\widetilde{v_p}$.

Set $\eta := s^{\frac{\alpha}{2}}$ and

$$\widetilde{w_p}(x,\eta) := \widetilde{v_p}(x,\eta^{\frac{2}{\alpha}}) = (\eta^{\frac{2}{\alpha}})^{1-\alpha}\widehat{u_p}(x,\eta^{\frac{2}{\alpha}}).$$

Then we have $\eta^2 \widetilde{w_p}(x,\eta) - a = (\Delta + p(x))\widetilde{w_p}$.

$\partial_t^2 w_p = (\Delta + p(x))w_p$ with $w(\cdot,0) = 0$ and $\partial_t w(\cdot,0) = a \Longrightarrow$

$\eta^2 \widehat{w_p}(x,\eta) - a = (\Delta + p(x))\widehat{w_p}$.

Therefore

$$\widetilde{w_p}(x,\eta) = (\eta^{\frac{2}{\alpha}})^{1-\alpha}\widehat{u_p}(x,\eta^{\frac{2}{\alpha}}) = \widehat{w_p}(x,\eta), \quad \eta > 0.$$

$u_p(x,t) = u_q(x,t)$ for $x \in \partial\Omega$ and $0 < t < t_0 \Longrightarrow u_p(x,t) = u_q(x,t)$ for $x \in \partial\Omega$ and $t > 0$
by analyticity in $t > 0$.
Laplace transform $\Longrightarrow \widehat{u_p}(x,s) = \widehat{u_q}(x,s)$ for $x \in \partial\Omega$ and $s > 0$.

Therefore $\widehat{w_p}(x,\eta) = \widehat{w_q}(x,\eta)$ for $x \in \partial\Omega$ and $\eta > 0 \Longrightarrow$
$w_p(x,t) = w_q(x,t)$ for $x \in \partial\Omega$ and $t > 0$ by Laplace inversion.

Carleman estimate method implies the uniqueness for inverse hyperbolic coefficient problem (Imanuvilov and Y. 2001)

However, the story is not finally very fantastic!

Disadvantage:

The conditions for observations are subject to hyperbolic equations.

- We have to choose sufficiently large portion of observation subboundary such as Lions subboundary condition and convexity, etc.
- Unhappier when we consider the inverse coefficient problem of determining $p(x)$ in

$$\partial_t^{\alpha}(u-a) = \mathrm{div}\,(p(x)\nabla u).$$

We need more restricted condition on $p(x)$, requiring that the wave speed is increasing inwards!

For fractional diffusion equations, we should be free from such spatial constraints and wave speed condition.

§10. Future aspects and conclusions

Future aspects.

I. More on approximate controllability. Fujishiro-Yamamoto (2014): approximate controllability is equivalent to the uniqueness in determining initial value!
exact controllability?

II. Fractional degenerate equation.
Memory effect by fractional derivative + spatial diffusion degeneracy = ???

III. Multi-term fractional derivatives, fractional derivative with relaxation factor, distributed fractional derivative motivated physically:

$$\sum_{k=1}^{N} q_k(x,t)\partial_t^{\alpha_k} v(x,t), \quad \int_0^t (t-s)^{-\alpha} g(t,s)\frac{\partial}{\partial s} v(x,s)ds, \quad \int_0^1 \theta(\alpha)\partial_t^{\alpha} v(x,t)d\alpha,$$

with $\theta \ge 0$ including linear combination of Dirac delta.
et cetera, et cetera, ...

Conclusions

Through Minicourse, I tried to

- show varieties of inverse problems for fractional diffusion-wave equations physically motivated.
- provide tool box for making mathematical analysis.
- demonstrate similarities to and differences from inverse problems for classical diffusion equations.
- suggest future meaningful works.

Thank you very much for your patience and interests!