On theoretical and numerical control and inverse problems for PDEs

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In collaboration with A. Doubova, M. González-Burgos, D.A. Souza, F. Maestre, I. Marín-Gayte and others . . .

Outline

Example: The Kermack-Mckendrick Model + Quarantine

$$\begin{cases} Q_t = p(t)S - \lambda(t)Q \\ S_t = -\beta \dfrac{I}{N} S - (p(t) + \rho(t))S + \lambda(t)Q \\ I_t = \beta \dfrac{I}{N} S - \gamma I \\ R_t = \gamma I + \rho(t)S \end{cases}$$

Q, S, I, R: **Q**uarantined, **S**usceptible, **I**nfectious and **R**ecovered individuals

$\lambda = \lambda(t)$: quarantine rate, $1/\lambda(t) =$ average time of confinement
$\rho = \rho(t)$: vaccinated individuals / time

Bi-objective optimal control problem:

- Goal 1: Minimize $J_1(\lambda, \rho) := \int_0^T I(t)\, dt$
- Goal 2: Minimize $J_2(\lambda, \rho) := |S(T) - S_T| + \int_0^T \rho(t)\, dt$

Q1: How can we choose λ and ρ?

Example: The Kermack-Mckendrick Model + Quarantine

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$\rho = \rho(t)$: vaccinated individuals / time

Controllability problem:

- Goal: Get $S(T) = S_d$, $I(T) = I_d$

Q2: How can we choose λ and ρ?

Example: The Kermack-Mckendrick Model + Quarantine

$$\begin{cases} Q_t = p(t)S - \lambda(t)Q \\ S_t = -\beta \dfrac{I}{N} S - (p(t) + \rho(t))S + \lambda(t)Q \\ I_t = \beta \dfrac{I}{N} S - \gamma I \\ R_t = \gamma I + \rho(t)S \end{cases}$$

Q, S, I, R: **Q**uarantined, **S**usceptible, **I**nfectious and **R**ecovered individuals

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Inverse problem: Now, λ and ρ are known but β and γ are not

Q3: Can we recover β and γ from **initial and final** values for S, I, R, Q?

Internal and boundary controllability of non-scalar parabolic systems

$$(1)\ \left\{ \begin{array}{l} y_t - \Delta y - Ay = Bv1_\omega \\ y = 0 \ \text{ on the boundary} \\ y|_{t=0} = y_0 \end{array} \right. \qquad (2)\ \left\{ \begin{array}{l} y_t - \Delta y - Ay = 0 \\ y = Bf1_\gamma \ \text{ on the boundary} \\ y|_{t=0} = y_0 \end{array} \right.$$

with

- $y = (y_1, \dots, y_n)^T$, $v = (v_1, \dots, v_m)^T$, $f = (f_1, \dots, f_m)^T$
- $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, $m < n$

Interesting case: $m << n$, for instance feeding very few species in a domain with many different populations

Controllability questions:
AC? Is (for instance) $\{y|_{t=T} : v \in L^2\}$ dense?
NC? Do we have (for instance) $\{y|_{t=T} : v \in L^2\} \ni 0$?

$$(1)\quad \begin{cases} y_{1,t} - \Delta y_1 - \displaystyle\sum_{j=1}^{n} A_{1,j} y_j = \sum_{k=1}^{m} B_{1,k} v_k 1_\omega \\ \dots \\ y_{n,t} - \Delta y_n - \displaystyle\sum_{j=1}^{n} A_{n,j} y_j = \sum_{k=1}^{m} B_{n,k} v_k 1_\omega \\ y_i = 0 \ \text{ on the boundary and} \\ y_i|_{t=0} = y_{i,0}, \quad i = 1, \dots, n \end{cases}$$

- $y = (y_1, \dots, y_n)^T$, $v = (v_1, \dots, v_m)^T$, $f = (f_1, \dots, f_m)^T$
- $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, $m < n$

Attention: the scalar systems are always AC and NC at any $T > 0$

$$\left\{ \begin{array}{l} z_t - \Delta z - az = v 1_\omega \\ z = 0 \\ z|_{t=0} = z_0 \end{array} \right. \qquad \left\{ \begin{array}{l} z_t - \Delta z - az = 0 \\ y = f 1_\gamma \\ z|_{t=0} = z_0 \end{array} \right.$$

Internal and boundary controllability of non-scalar parabolic systems

$$(1)\ \begin{cases} y_t - \Delta y - Ay = Bv1_\omega \\ y = 0 \\ y|_{t=0} = y_0 \end{cases} \qquad (2)\ \begin{cases} y_t - \Delta y - Ay = 0 \\ y = Bf1_\gamma \\ y|_{t=0} = y_0 \end{cases}$$

N: spatial dimension, n: number of states, m: number of controls

Notation: $[P;R] := [R|PR|\cdots|P^{d-1}R]$ for $R \in \mathbf{R}^{d\times r}$, $P \in \mathbf{R}^{d\times d}$

Known results:

- (1) NC $\Leftrightarrow$ $\mathrm{rank}\,[A;B] = n$ (Kalman)
- For $N = 1$: (2) NC $\Leftrightarrow$ $\mathrm{rank}\,[L_k;B_k] = nk \ \ \forall k \geq 1$
  Here: $B_k := [B\ldots B]^T$, $L_k := \mathrm{diag}\,(L_1,\ldots,L_k)$, $L_j = \lambda_j \mathrm{Id.} - A$
  [Ammar-Khodja et al. 2010]

  In particular, if $N = 1$, $m = 1$ (1D in space, one control):

  $$(2)\ \text{NC} \ \Leftrightarrow\ \mathrm{rank}\,[A;B] = n,\ \ \mu_i - \mu_j \neq \lambda_k - \lambda_\ell \ \text{ for } \ (k,i) \neq (\ell,j)$$

Internal and boundary controllability of non-scalar parabolic systems

$$(1)\ \begin{cases} y_t - \Delta y - Ay = Bv1_\omega \\ y = 0 \\ y|_{t=0} = y_0 \end{cases} \qquad (2)\ \begin{cases} y_t - \Delta y - Ay = 0 \\ y = Bf1_\gamma \\ y|_{t=0} = y_0 \end{cases}$$

Problem 1: For $N \geq 2$, $m < n$: results for (2) are unknown
Problem 2: For variable A and/or $y_t - D\Delta y - Ay = \dots$: general criteria?

Null controllability of the free-boundary two-phase Stefan problem

$$(NC)_1 \quad \begin{cases} y_t - d_l y_{xx} = 0, \ x \in (0, \ell(t)), \ t \in (0, T) \\ z_t - d_r z_{xx} = 0, \ x \in (\ell(t), L), \ t \in (0, T) \\ \\ y|_{x=0} = v_l, \quad z|_{x=L} = v_r, \quad t \in (0, T) \\ \\ (d_l y_x - d_r z_x)|_{x=\ell(t)} = -k\ell'(t), \quad t \in (0, T) \\ \dots \end{cases}$$

$$(NC)_2 \quad \begin{cases} y(x, T) = 0, \quad x \in (0, \ell(T)), \quad z(x, T) = 0, \quad x \in (\ell(T), L) \\ \ell(T) = \ell_T \end{cases}$$

Two-phase Stefan problem

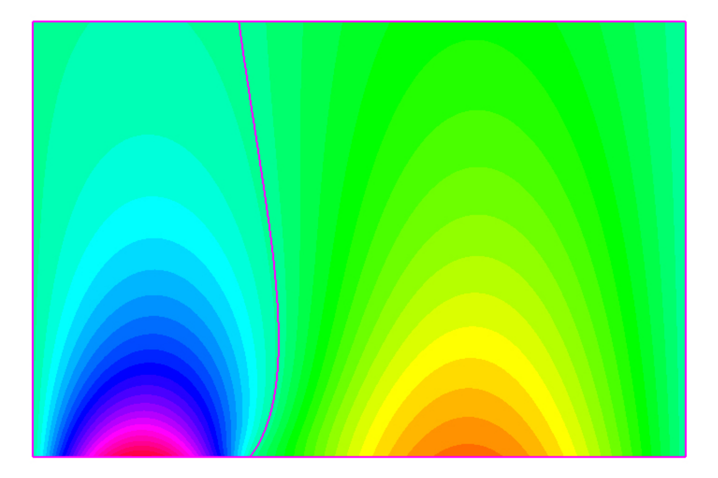

Figure: Uncontrolled solution

Two-phase Stefan problem

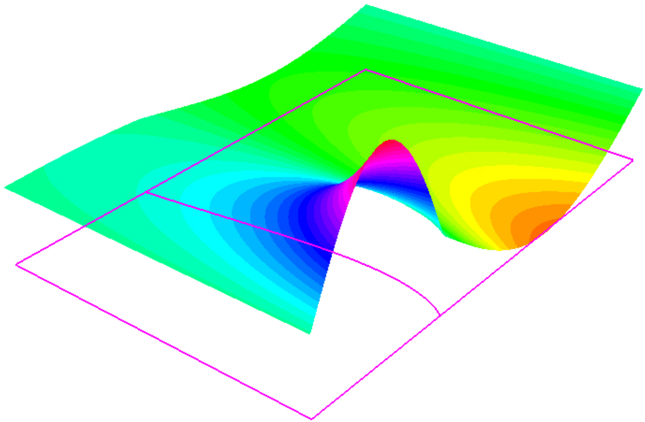

Figure: Uncontrolled solution

$$(NC)_1 \qquad \begin{cases} y_t - d_l y_{xx} = 0, \;\; x \in (0, \ell(t)), \;\; t \in (0, T) \\ z_t - d_r z_{xx} = 0, \;\; x \in (\ell(t), L), \;\; t \in (0, T) \\ \dots \\ (d_l y_x - d_r z_x)|_{x=\ell(t)} = -k\ell'(t), \;\; t \in (0, T) \\ \dots \end{cases}$$

$$(NC)_2 \qquad \begin{cases} y(x, T) = 0, \;\; x \in (0, \ell(T)), \quad z(x, T) = 0, \;\; x \in (\ell(T), L) \\ \ell(T) = \ell_T \end{cases}$$

Results (with D.A. Souza and others):

- Local NC, i.e. $\exists \varepsilon > 0$ such that $\|y_0\|_{H_0^1} + \|z_0\|_{H_0^1} + |\ell_0 - \ell_T| \leq \varepsilon \Rightarrow \exists v_l, v_r, \ell, y, z$ satisfying $(NC)_1, (NC)_2$
- Computations

Two-phase Stefan problem

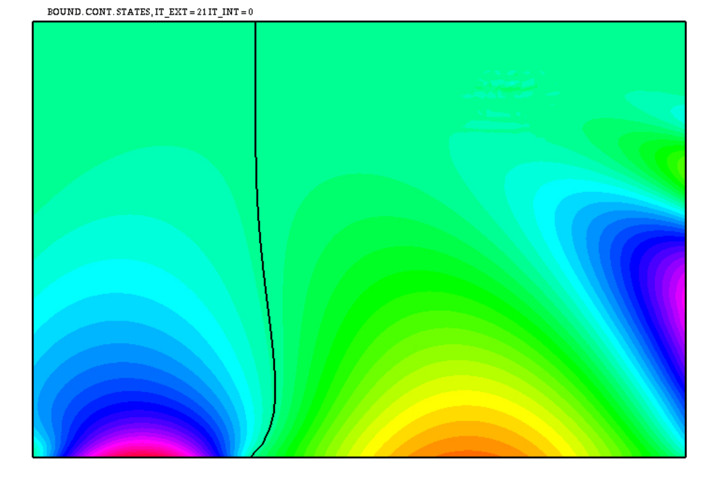

Figure: Controlled solution

Two-phase Stefan problem

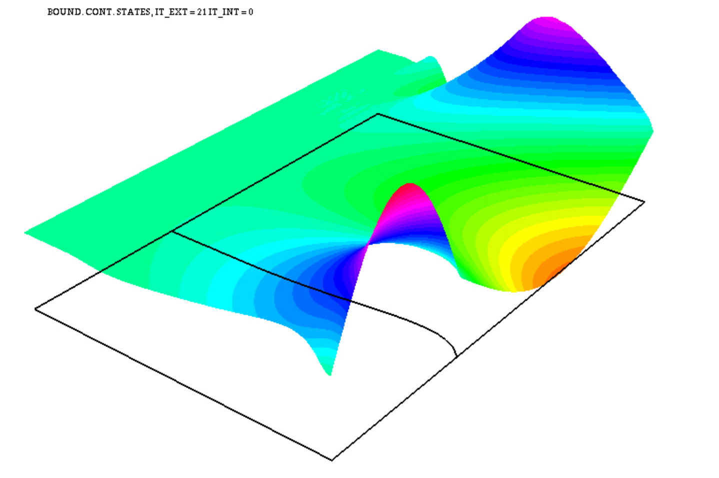

Figure: Controlled solution

Two-phase Stefan problem

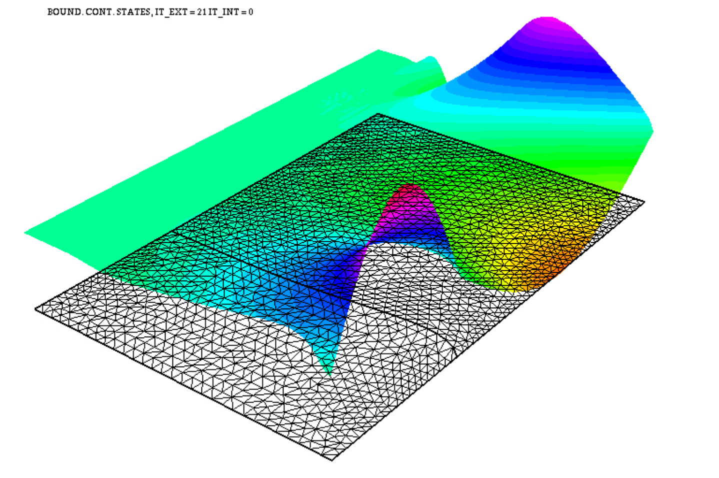

Figure: Controlled solution and final mesh

Two-phase Stefan problem

- **Problem 3:** Results with one boundary control?
- **Problem 4:** Global controllability?
- **Problem 5:** Local (and global) results for semilinear and nonlinear PDEs?
- **Problem 6:** Results for higher spatial dimensions? For instance, domains close to a ball?

Elastography:

A technique to detect elastic properties of tissue

Mathematical model components:

- The system (displacements, acoustic waves generator, MR or ultrasound);

$$\begin{cases} u_{tt} - \nabla \cdot \left(\mu(\nabla u + \nabla u^T) + \lambda(\nabla \cdot u)\mathrm{Id.} \right) = f(x,t) & \text{in } \Omega \times (0,T) \\ u = \varphi & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega \end{cases}$$

- The observation (stress captor):

$$\Lambda := \sigma(u) \cdot n = \left(\mu(\nabla u + \nabla u^T) + \lambda(\nabla \cdot u)\mathrm{Id.} \right) \cdot n \quad \text{on } \gamma \times (0,T)$$

Start: (u_0, u_1), applying: φ on $\partial\Omega \times (0,T)$, measuring: Λ on $\gamma \times (0,T)$

- The inverse problem: find μ and λ (stiffness quantification) from Λ

A Calderón-like IP — Uniqueness? Stability? Reconstruction?

Figure: An elastogram for a glioblastoma (brain tumor)

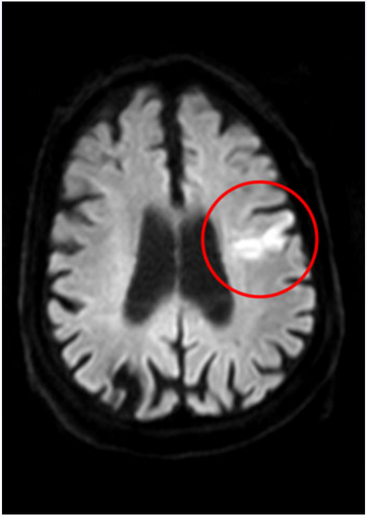

Figure: Arteriosclerosis (thickening, hardening and loss of elasticity) in the carotid arteria. Diagnosis by MRI

$$\begin{cases} u_{tt} - \nabla\cdot\left(\mu(\nabla u + \nabla u^T) + \lambda(\nabla\cdot u)\mathrm{Id.}\right) = f(x,t) & \text{in } \Omega\times(0,T) \\ u = \varphi & \text{on } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega \end{cases}$$

Reconstruction (main result, with F. Maestre):
Assume $f, f_t \in L^2(Q)^N$, $u_0 = 0$, $u_1 \in H_0^1(\Omega)^N$, $\Lambda \in L^2(\Sigma)^N$
Introduce a related (direct) extremal problem ($R > 0$ is given):

$$\begin{cases} \text{Minimize } \ I(\mu,\lambda) \\ \text{Subject to } \ (\mu,\lambda) \in \mathbb{K}(R) \end{cases}$$

$$I(\mu,\lambda) := \frac{1}{2}\int_0^T \left\|\sigma(u)\cdot n\big|_\gamma - \Lambda\right\|^2 dt$$

$$\mathbb{K}(R) := \{\, (\mu,\lambda) \in \mathbb{BV}(\Omega), \ \ \alpha \le \mu, \lambda \le \beta, \ \ TV(\mu), TV(\lambda) \le R \,\}$$

Then: $\forall R > 0$ $\exists$ at least one solution (μ_R, λ_R)

$$\begin{cases} u_{tt} - \nabla \cdot \left(\mu(\nabla u + \nabla u^T) + \lambda(\nabla \cdot u)\mathrm{Id.} \right) = f(x,t) & \text{in } \Omega \times (0,T) \\ u = \varphi & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega \end{cases}$$

In other words:

Under the assumptions $\alpha \leq \mu, \lambda \leq \beta$ and $TV(\mu), TV(\lambda) \leq R$, we can compute μ and λ from Λ

Problem 7: Results with no restriction on TV?

A numerical experiment

The domain and the mesh

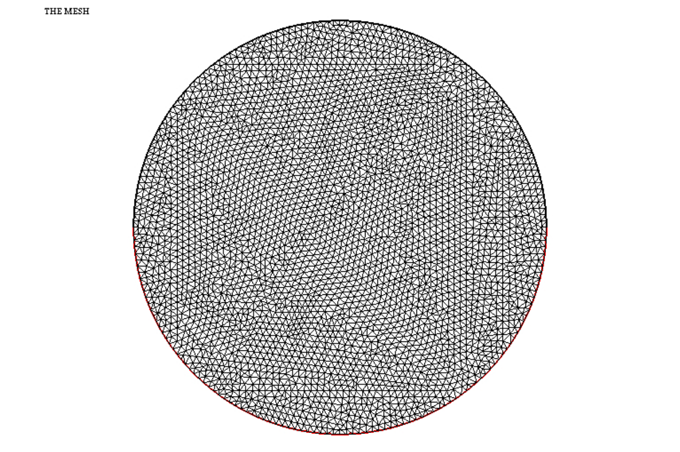

Figure: Number of nodes: 3629 – Number of triangles: 7056

TEST 1

Starting: $\mu = 5$ Target: $\mu = 10$ in D, $\mu = 1$ outside. Same λ

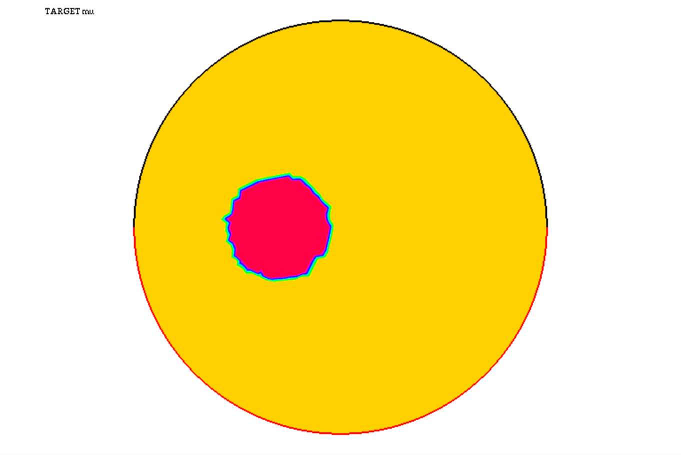

Figure: The target μ. The information Λ is taken accordingly

The algorithm: Augmented Lagrangian + L-BFGS
(limited memory quasi-Newton, Broyden, Fletcher, Goldfarb and Shanno)
Final cost $\sim 9.6 \times 10^{-8}$, 158 comp. of the cost, 78 comp. of the gradient.

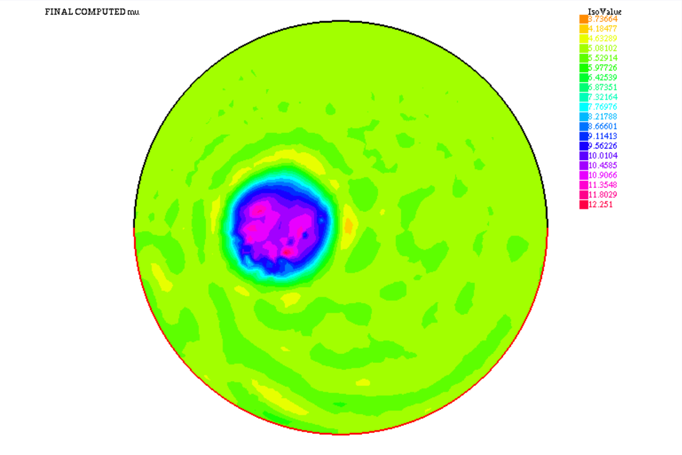

Figure: The computed μ

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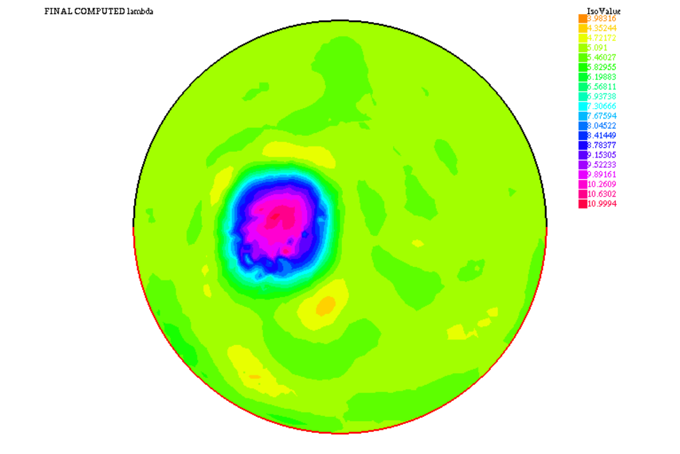

Figure: The computed λ

TEST 2

Starting: $\mu = 5$ Target: $\mu = 10$ in $D_1 \cup D_2$, $\mu = 1$ outside. Same λ

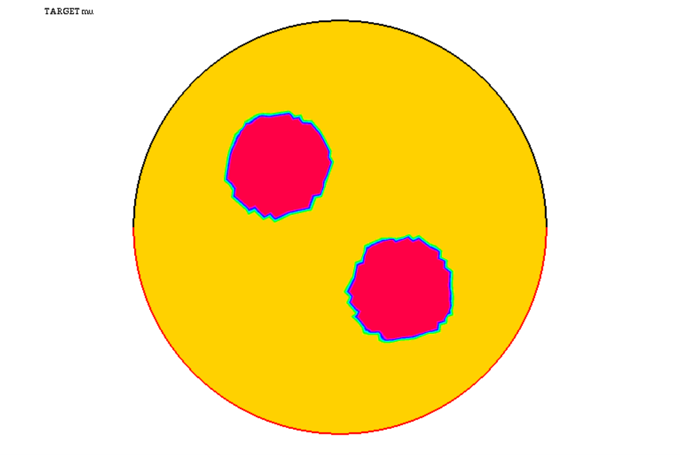

Figure: The target μ. The information Λ is taken accordingly

The algorithm: Augmented Lagrangian + L-BFGS
Final cost $\sim 9.6 \times 10^{-8}$, 180 comp. of the cost, 80 comp. of the gradient.

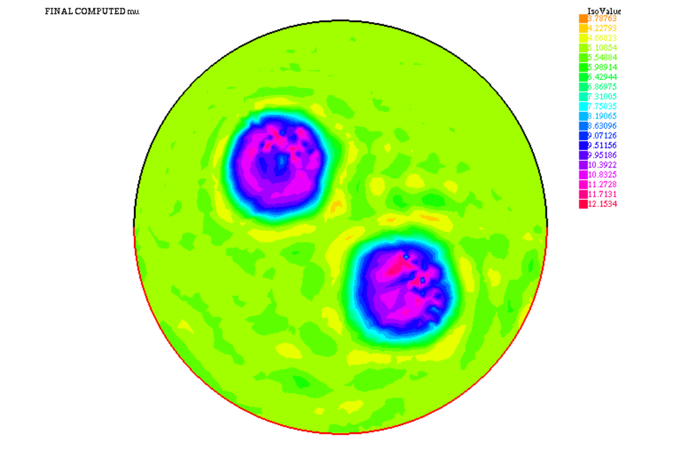

Figure: The computed μ

The algorithm: Augmented Lagrangian + L-BFGS
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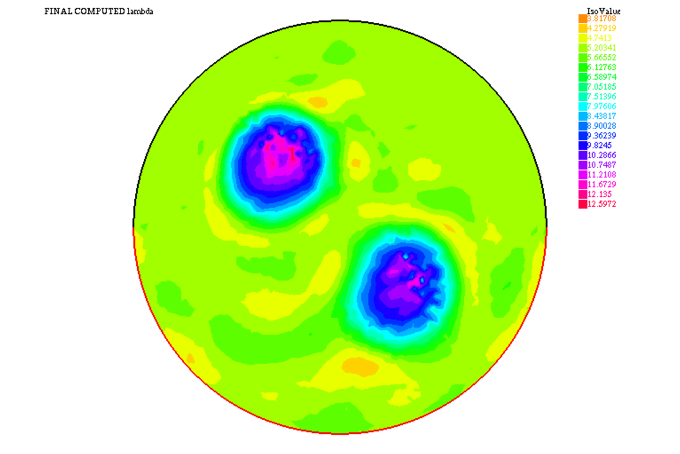


Figure: The computed λ

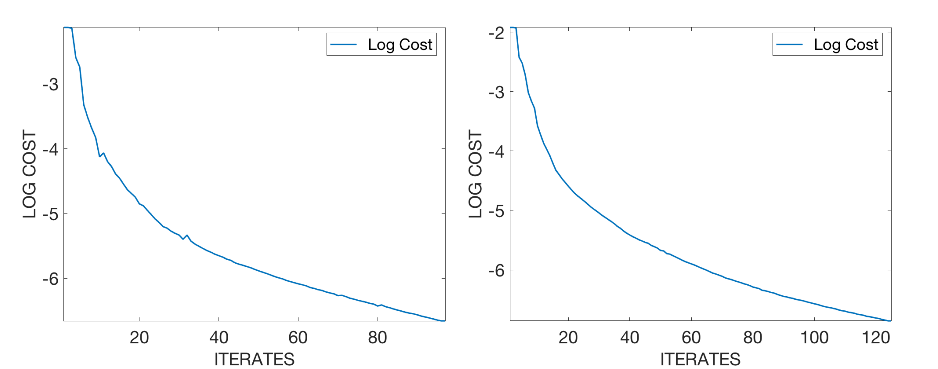


Figure: $\log$ of the cost versus number of iterates. Case 1 (left) and Case 2 (right).

Navier-Stokes-like systems

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}1_\omega, \quad \nabla\cdot\mathbf{u} = 0 \\ \text{Boundary and initial conditions} \end{cases}$$

Existence: $\forall\, \mathbf{f}, \mathbf{u}_0$ in reasonable spaces $\exists(\mathbf{u}, p)$ (unique if $N = 2$)

$\exists$ many reasons to consider related control problems:
optimum design, optimal suction problems, pollution minimization, etc.

Bi-objective control problems and stationary Navier-Stokes

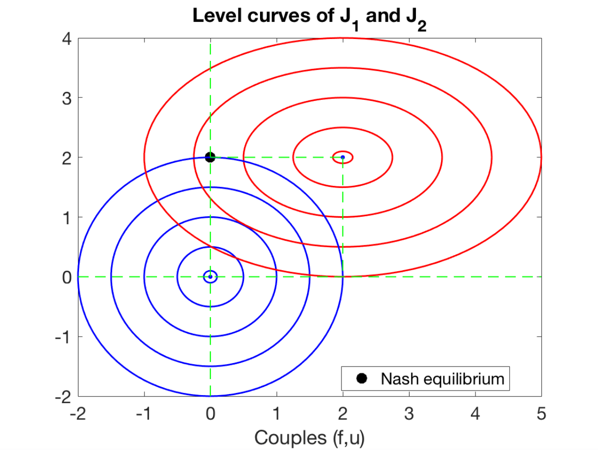
$$\begin{cases} E_1(\mathbf{u}, p) = \mathbf{f}_1 1_{\omega_1} + \mathbf{f}_2 1_{\omega_2} \text{ in } \Omega \\ E_2(\mathbf{u}) = 0 \text{ in } \Omega \\ \dots \end{cases}$$

$$J_i(\mathbf{f}_1, \mathbf{f}_2, \mathbf{u}, p) = \frac{a}{2} \int_{O_i} |\mathbf{u} - \mathbf{u}_{id}|^2 + \frac{\mu}{2} \int_{\omega_i} |\mathbf{f}_i|^2 \quad i = 1, 2$$

"Minimizing" J_1 and J_2?
We look for Nash equilibria $(\mathbf{f}_1, \mathbf{f}_2)$:

$$\begin{cases} J_1(\mathbf{f}_1, \mathbf{f}_2, \mathbf{u}, p) \le J_1(\mathbf{f}_1', \mathbf{f}_2, \mathbf{y}, q) \quad \forall \mathbf{f}_1' \\ J_2(\mathbf{f}_1, \mathbf{f}_2, \mathbf{u}, p) \le J_2(\mathbf{f}_1, \mathbf{f}_2', \mathbf{y}, q) \quad \forall \mathbf{f}_2' \end{cases}$$

An illustration of bi-objective extremal problems and Nash equilibria



$$\begin{cases} E_1(\mathbf{u}, p) = \mathbf{f}_1 1_{\omega_1} + \mathbf{f}_2 1_{\omega_2} \ \text{ in } \Omega \\ E_2(\mathbf{u}) = 0 \ \text{ in } \Omega \\ \dots \end{cases}$$

$$J_i(\mathbf{f}_1, \mathbf{f}_2, \mathbf{u}, p) = \frac{a}{2} \int_{O_i} |\mathbf{u} - \mathbf{u}_{id}|^2 + \frac{\mu}{2} \int_{\omega_i} |\mathbf{f}_i|^2 \quad i = 1, 2$$

Results (with I. Marín-Gayte):

- $\exists$ (delicate ...)
- Characterization: $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{u}, p)$ Nash equilibrium $\Rightarrow$ $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{u}, p)$ Nash quasi-equilibrium, i.e.
  $\exists(\mathbf{w}_i, q_i)$ such that

$$\begin{cases} E_1(\mathbf{u}, p) = \mathbf{f} 1_{\omega} \ \text{ in } \Omega \\ E_2(\mathbf{u}) = 0 \ \text{ in } \Omega \\ E_1^*(\mathbf{w}_i, q_i) = (\mathbf{u} - \mathbf{u}_{id}) 1_{O_i} \ \text{ in } \Omega \\ E_2(\mathbf{w}_i) = 0 \ \text{ in } \Omega \\ \mathbf{f}_i = -\frac{a}{\mu} \mathbf{w}_i \big|_{\omega_i} \\ \dots \end{cases}$$

- Computation

A numerical experiment: control in a channel

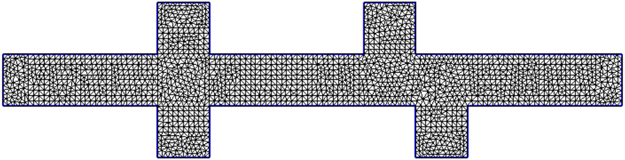


Figure: The domain and a "rough" mesh; Ω is composed of the main pipe, two first rectangles (ω_1 and ω_2), a second upper rectangle $\mathcal{O}_1$ and a second lower rectangle $\mathcal{O}_2$. Number of nodes: 1541. Number of triangles: 2774.

A numerical experiment: control in a channel

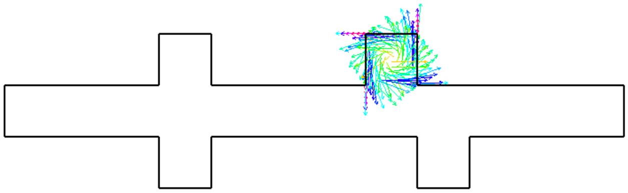


Figure: The function u_{1d}; $u_{2d} = 0$ (recall: $J_i = \frac{a}{2} \int_{O_i} |\mathbf{u} - \mathbf{u}_{id}|^2 + \frac{\mu}{2} \int_{\omega_i} |\mathbf{f}_i|^2$).

A numerical experiment: control in a channel

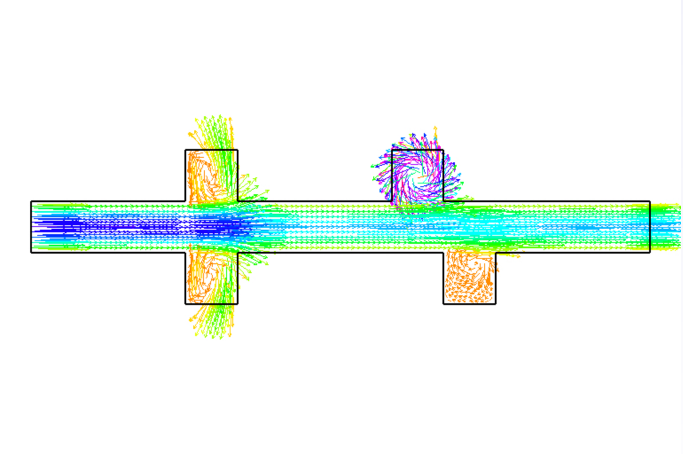


Figure: The final computed velocity fields (Newton) for $Re = 1200$ and $a = 1.99$, $\mu = 0.01$ (recall: $J_i = \frac{a}{2} \int_{O_i} |\mathbf{u} - \mathbf{u}_{id}|^2 + \frac{\mu}{2} \int_{\omega_i} |\mathbf{f}_i|^2$).

Other results:

- $\exists$, characterization and computation of other equilibria (Pareto)
- The same for time-dependent problems: linear and semilinear heat, wave, etc.
- Hierarchical control: Stackleberg-Nash, Stackleberg-Pareto, Pareto-Stackleberg, ...
  For instance:
  $$\begin{cases} y_t - y_{xx} = f 1_\omega + v 1_{\mathcal{O}} \\ \text{Bound. and initial conditions} \end{cases}$$
  1. The substep (optimal control):
  $$\text{For any } v \text{ find } f(v) \text{ minimizing } J = J(f, y; v)$$
  2. The NC Stackelberg problem:
  $$\text{Find } v \text{ fsuch that } y|_{t=T} = 0$$

  $\exists$, Characterization, Computation

Problem 8: Results for time-dependent Navier-Stokes-like systems?

Navier-Stokes, Dirichlet

$$\begin{cases} \textbf{Navier-Stokes PDEs} & (\mathbf{x}, t) \in \Omega \times (0, T) \\ \mathbf{u} = 0, \quad \mathbf{x} \in \partial\Omega \setminus \Gamma, \quad t \in (0, T) \\ \mathbf{u} = \mathbf{f} 1_{\Gamma}, \quad \mathbf{x} \in \Gamma, \quad t \in (0, T) \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 \end{cases}$$

Conjecture [JL Lions, 90]

AC: $\forall \mathbf{u}_0, \mathbf{u}_T, \ \forall \varepsilon > 0, \ \ \exists \mathbf{f}$ such that $\|\mathbf{u}(\cdot\,, T) - \mathbf{u}_T\| \leq \varepsilon$
NC: $\forall \mathbf{u}_0 \ \ \exists \mathbf{f}$ such that $\mathbf{u}(\cdot\,, T) = 0$

Many partial (positive) results — Among them:

(1) Local NC, also for large T (Dirichlet and other BC's)
(2) Global NC (all-boundary control, $\Gamma = \partial\Omega$)
(3) Global NC (Navier-slip-2D, periodic or no boundary), etc.

Problem 9: Global AC? Global NC?

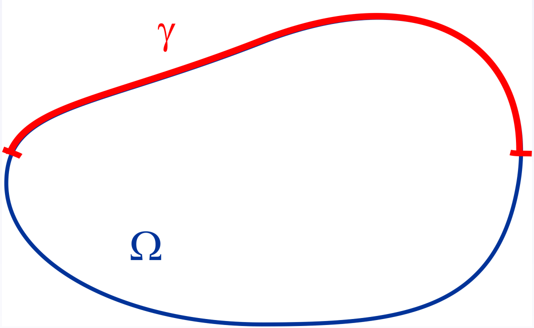


Figure: The domain and the active boundary

Navier-Stokes, Navier-slip-with-friction

$$\begin{cases} \textbf{Navier-Stokes PDEs} \quad (\mathbf{x}, t) \in \Omega \times (0, T) \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad [2\nu D\mathbf{u}\,\mathbf{n} + M\mathbf{u}]_{\mathrm{tan}} = 0, \quad \mathbf{x} \in \partial\Omega \setminus \Gamma, \quad t \in (0, T) \\ \mathbf{u} = \mathbf{f} 1_\Gamma, \quad \mathbf{x} \in \Gamma, \quad t \in (0, T) \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 \end{cases}$$

$M = M(\mathbf{x}, t)$: smooth and symmetric
The fluid slips, normal stresses are “proportional” to tangential $\mathbf{u}$

Theorem [Coron-Marbach-Sueur, 2018]

Global NC, i.e. $\forall \mathbf{u}_0 \in H_\Gamma \;\; \exists f$ with $\mathbf{u}(\mathbf{x}, T) \equiv 0$

Also global ECT: $\forall$ admiss. trajectory $\exists \mathbf{f}$ with $\mathbf{u}(\mathbf{x}, T) \equiv \mathbf{u}_*(\mathbf{x}, T)$

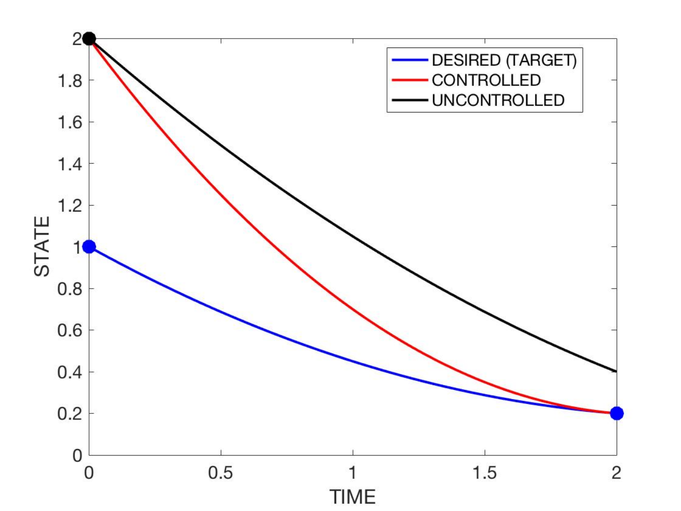


Figure: Exact controllability to the trajectories (illustration)

For the proof of Coron-Marbach-Sueur's result:

- It suffices: to reach arbitrarily small states
- Extension: $\Omega \to \mathcal{O}$ and distributed control
- Change of scale: $t = \varepsilon t'$, $\mathbf{u} = \varepsilon^{-1}\mathbf{u}'$, etc.
  Now: new $\mathbf{u}'_0$ and ν' are small and new T' is large
- $(\mathbf{u}', p', \xi', \sigma') = \mathbf{U}^0 + \varepsilon\mathbf{U}^1 + \varepsilon\mathbf{U}^\varepsilon + (\sqrt{\varepsilon}\,\mathbf{v}(\mathbf{x}, t'; \frac{\varphi(\mathbf{x})}{\sqrt{\varepsilon}}), 0, 0, 0) +$ STT's
  - $\mathbf{U}^0$ such that $\mathbf{u}^0|_{t=0} = 0$, $\nabla \times \mathbf{u}^0 = 0$ and $\mathbf{u}^0(\mathbf{x}, t') = 0 \ \ \forall t' > t'(\mathbf{x}) \ \ \forall \mathbf{x} \in \mathcal{O}$
  - $\mathbf{U}^1$ such that $\mathbf{u}^1|_{t=0} = \mathbf{u}'_0$, $\|\mathbf{u}^1|_{t'=T/2}\| \leq C\varepsilon$
  - $\mathbf{U}^\varepsilon$ such that $\|\mathbf{u}^\varepsilon|_{t'=3T/4}\| \leq C\varepsilon^{1/2}$
  - $\mathbf{v} = \mathbf{v}(\mathbf{x}, t'; z)$ is the solution to a Prandtl-like PDE, with source ξ_v
- After some work: $\mathbf{v}|_{t'=\varepsilon T}$ is "prepared" with ξ_v (Possible !!!)
  Hence, $\mathbf{v}|_{t'=T'}$ is small

Adaptation to Dirichlet conditions?

Also: global NC and global ECT for Boussinesq

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = 0, \quad \nabla\cdot\mathbf{u} = 0 \\ \theta_t + \mathbf{u}\cdot\nabla\theta - \kappa\Delta\theta = 0 \end{cases}$$

with

$$\begin{cases} \mathbf{u}\cdot\mathbf{n} = 0, \quad [2\nu D\mathbf{u}\,\mathbf{n} + M\mathbf{u}]_{\rm tan} = 0, \quad \dfrac{\partial\theta}{\partial n} + m\theta = 0, \quad \mathbf{x}\in\partial\Omega\setminus\Gamma \\ \mathbf{u} = \mathbf{f}1_\Gamma, \quad \theta = h1_\Gamma, \quad \mathbf{x}\in\Gamma \end{cases}$$

(results with Chaves-Silva, Le Balch and Souza 2022)

Other questions:

- **Problem 10:** Navier-Stokes + dynamic boundary conditions?

$$\begin{cases} \dots \\ \mathbf{u}\cdot\mathbf{n} = 0, \;\; [\mathbf{u}_t + 2\nu D\mathbf{u}\,\mathbf{n} + M\mathbf{u}]_{\tan} = 0, \;\; \mathbf{x}\in\partial\Omega\setminus\Gamma, \;\; t\in(0,T) \\ \mathbf{u} = \mathbf{f}1_\Gamma, \;\; \mathbf{x}\in\Gamma, \;\; t\in(0,T) \end{cases}$$

- **Problem 11:** Variable density Navier-Stokes or Boussinesq?

$$\begin{cases} \rho_t + \mathbf{u}\cdot\nabla\rho = 0 \\ \rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla p = 0, \;\; \nabla\cdot\mathbf{u} = 0 \\ \dots \end{cases}$$

- **Problem 12:** Boussinesq viscous heat sources + Navier-slip and Robin?

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = 0, \;\; \nabla\cdot\mathbf{u} = 0 \\ \theta_t + \mathbf{u}\cdot\nabla\theta - \kappa\Delta\theta = 2\nu D\mathbf{u} : \nabla\mathbf{u} \\ \dots \end{cases}$$

Results for linearized Oldroyd systems

$$\begin{cases} u_t - \Delta u + \nabla p = \nabla \cdot \tau, \quad \nabla \cdot u = 0 \\ \tau_t + a\tau = bDu \\ u = f 1_\Gamma \ \text{ on } \ \partial\Omega \times (0, T) \\ + \dots \end{cases}$$

Navier-Stokes + PDE for τ (elastic tensor), linearized system
Particle interaction: inertia + friction (viscosity) + memory (elasticity)

¿Boundary control?

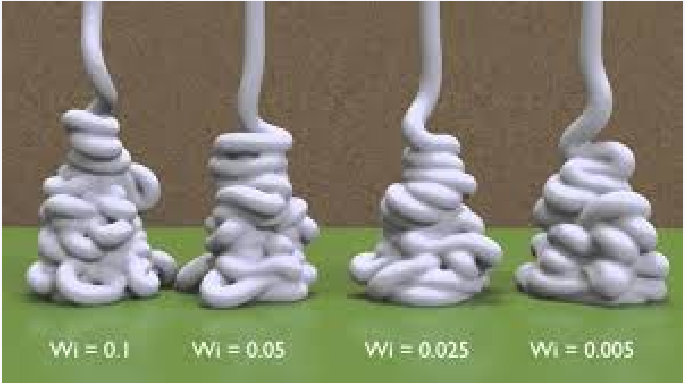


Figure: A visco-elastic fluid

Figure: A visco-elastic fluid

$$\begin{cases} \text{Linearized Oldroyd} \\ u = f 1_\Gamma \ \text{ on } \ \partial\Omega \times (0, T) \\ u|_{t=0} = u_0 + \dots \end{cases}$$

u_0 is given. ¿AC? ¿$\forall \varepsilon > 0$ $\exists f_\varepsilon$ with $\| u|_{t=T} \| \leq \varepsilon$?

¿NC? ¿$\exists f$ with $u|_{t=T} = 0$?

Results (with A. Doubova, D.A. Souza and others):

- AC holds
- In general, NC does not

Problem 13: Computation of f_ϵ (?)
Problem 14: What about the original nonlinear problem?

$$\begin{cases} u_t + (u \cdot \nabla) u - \Delta u + \nabla p = \nabla \cdot \tau, \quad \nabla \cdot u = 0 \\ \tau_t + (u \cdot \nabla) \tau + a\tau + g(\nabla u, \tau) = bDu \\ + \dots \end{cases}$$

THANK YOU VERY MUCH . . .