Nonlocal basis pursuit: Nonlocal optimal design of conductive domains in the vanishing material limit

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Optimal design in conduction

A typical optimal design problem in conductivity is the following:

$$\max_{\kappa \in \mathbb{A}} \min_{H_0^1(\Omega)} \left[\frac{1}{2} \int_\Omega \kappa(x) |\nabla u(x)|^2 \, dx - \int_\Omega f(x) u(x) \, dx \right]$$

where $\Omega \subset \mathbb{R}^n$ a bounded, open domain, $\gamma \in (0,1)$ and

$$\mathbb{A} = \left\{ \kappa \in L^\infty(\Omega) \; : \; \kappa(x) \in [\underline{\kappa}, \bar{\kappa}], \int_\Omega \kappa(x) \, dx \leq \gamma |\Omega| \right\}$$

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This is max-min of *compliance maximization* in optimal design of conducting materials, using the Dirichlet's principle.

Optimal design in conduction

Alternatively to using Dirichlet's principle we could use that of Kelvin for the fluxes, where the heat flux is given by the variational principle

$$q = \operatorname{argmin}_{p \in \mathbb{Q}(f)} I_{loc}(\kappa; p),$$

with

$$I_{loc}(\kappa; p) = \frac{1}{2} \int_{\Omega} \kappa^{-1}(x) |p(x)|^2 \, dx,$$

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$$\mathbb{Q}(f) = \{ q \in H(\text{div}, \Omega, \mathbb{R}^n) \; : \; \text{div}(q) = f \} \, .$$

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Due to joint-convexity of the functional I_{loc}, this problem attains its minimum.

Vanishing material fraction limit

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This problem has a **basis pursuit** structure: we look for the sparsest solution in the L^1 sense of the underdetermined system of PDE's.

Vanishing material fraction limit

This problem has been studied in the **context of elasticity**:

- In the pioneering paper by Michell (*The limits of economy of material in frame-structures*. Phil. Mag. 1904): Michell trusses.
- By Allaire and Kohn (*Optimal design of minimum weight and compliance in place stress using extrema microstructures* Eur. J. Mech. A Solids, 1993): which formally obtained the vanishing material limit in elasticity
- Olbermann (*Michell trusses in two dimensions as a Γ-limit of optimal design in linear elasticity*. Cal. Var. 2017): studied the previous formal result in the context of Γ-convergence
- Bouchitté, Gangbo, Seppecher (*Michell trusses and lines of principal action*. M3AS, 2008): study the relaxed problem, obtaining properties of optimal measures
- Evgrafov, Sigmund (*Sparse basis pursuit for compliance minimization in the vanishing volume ratio limit*, ZAMM, 2020): Attempt of numerical approximation of the limit problem

Nonlocal optimal design

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Nonlocal kernel: a radial non-negative function w_δ with support in $B(0,\delta)$ such that $\int_{\mathbb{R}^n} |x|^2 w_\delta^2(x)\, dx = K_{2,n}$.

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NONLOCAL OPERATORS:

- Nonlocal two-points gradient of u:

$$\tilde{\mathcal{G}}_\delta u(x,x') = |u(x) - u(x')| w_\delta(x,x')$$

- Nonlocal divergence of $\tilde{q}$: $\mathcal{D}_\delta$ is the negative adjoint of $\tilde{\mathcal{G}}_\delta$,

$$(\mathcal{D}_\delta \tilde{q}, v)_{L^2(\Omega)} = -(\tilde{q}, \tilde{\mathcal{G}}_\delta v)_{L^2(\Omega_\delta,\Omega_\delta)}, \quad \forall\, \tilde{q},\, v$$

Nonlocal optimal design

We are interested in a nonlocal Kelvin's principle for fluxes:

$$\min_{\mathbb{Q}_\delta(f)} I(\tilde{\kappa}, \tilde{q}),$$

with

$$\mathbb{Q}_\delta(f) = \left\{ \tilde{q} \in L^2(\Omega_\delta \times \Omega_\delta) \ : \ \mathcal{D}_\delta \tilde{q} = f \right\},$$

and

$$I(\tilde{\kappa}, \tilde{q}) = \frac{1}{2} \int_{\Omega_\delta} \int_{\Omega_\delta} \tilde{\kappa}^{-1}(x, x') |\tilde{q}(x, x')|^2 \, dx \, dx'.$$

Nonlocal optimal design

Nonlocal optimal design for fluxes

The following nonlocal optimal design problem admits solutions:

$$\min_{(\kappa,\tilde{q})\in\mathbb{A}_\delta\times\mathbb{Q}_\delta(f)} I(\tilde{\kappa},\tilde{q}),$$

where given

$$\kappa\in\mathbb{A}_\delta=\left\{\kappa\in L^\infty(\Omega_\delta)\ :\ \kappa(x)\in[\underline{\kappa},\bar{\kappa}],\ \int_{\Omega_\delta}\kappa(x)\,dx\leq\gamma|\Omega_\delta|\right\},$$

$\tilde{\kappa}$ is defined as the parametrization (through *harmonic averaging*)

$$\tilde{\kappa}=2\kappa(x)\kappa(x')[\kappa(x)+\kappa(x')]^{-1}.$$

Nonlocal optimal design

THEOREM: Limit as $\delta \searrow 0$

Nonlocal optimal design problem

$$\min_{(\kappa,\tilde{q})\in\mathbb{A}_\delta\times\mathbb{Q}_\delta(f)} I(\tilde{\kappa},\tilde{q}),$$

Γ-converges as $\delta \searrow 0$ to the local optimal design problem

$$\min_{(\kappa,q)\in\mathbb{A}\times\mathbb{Q}(f)} I_{loc}(\kappa;q).$$

Vanishing volume fraction limit in the nonlocal case

Lebesgue spaces with mixed exponents (Benedek, Panzone, *Duke Math. J.*, 1961): $L^{p,q}(\Omega_\delta \times \Omega_\delta)$, $1 \le p, q \le +\infty$, is the space of measurable functions $\tilde{q} : \Omega_\delta \times \Omega_\delta \to \mathbb{R}$ such that

$$\|\tilde{q}\|_{L^{p,q}(\Omega_\delta \times \Omega_\delta)} = \left\{ \int_{\Omega_\delta} \left[\int_{\Omega_\delta} |\tilde{q}(x,x')|^q \, dx' \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} < +\infty.$$

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We denote $L^{p,q}_a(\Omega_\delta \times \Omega_\delta)$ the subspace of antisymmetric ($\tilde{q}(x, x') = -\tilde{q}(x', x)$) functions in $L^{p,q}(\Omega_\delta \times \Omega_\delta)$.

Vanishing volume fraction limit in the nonlocal case

Formal limit as $\gamma \searrow 0$

The formal limit as $\gamma \searrow 0$ of

$$\min_{(\kappa,\tilde{q})\in\mathbb{A}_\delta\times\mathbb{Q}_\delta(f)} I(\tilde{\kappa},\tilde{q}),$$

where

$$\inf_{\bar{\mathbb{Q}}^a_\delta(f)} \|\tilde{q}\|_{L^{1,2}(\Omega_\delta\times\Omega_\delta)},$$

is

$$\bar{\mathbb{Q}}^a_\delta(f) = \left\{\tilde{q} \in L^{1,2}_a(\Omega_\delta \times \Omega_\delta) \ : \ \mathcal{D}_\delta(\tilde{q}) = f\right\}.$$

Vanishing volume fraction limit in the nonlocal case

Theorem

The unit ball $B_{L_a^{1,2}}(\Omega_\delta \times \Omega_\delta)$ is sequentially compact with respect to biting convergence in $L^{1,2}(\Omega_\delta \times \Omega_\delta)$, i.e, any sequence $\tilde{q}_k$ admits a subsequence (still denoted the same) such that there exists a non-increasing sequence of sets $E_m \subset \Omega_\delta$ with measures converging to zero, such that $\tilde{q}_k$ is weakly convergent in $L^{1,2}((\Omega_\delta \backslash E_m) \times \Omega_\delta)$ for any $m \geq 1$.

Theorem

In high constrast with local problem, basis pursuit nonlocal problem

$$\inf_{\bar{\mathbb{Q}}_\delta^a(f)} \|\tilde{q}\|_{L^{1,2}(\Omega_\delta \times \Omega_\delta)},$$

admits solution

Recovering of the local basis pursuit problem

Partial approximation result

$$\inf_{\operatorname{div}(q)=f} \int_\Omega |q(x)|\, dx \leq \liminf_{\delta \searrow 0} \left[\inf_{\bar{\mathbb{Q}}^a_\delta(f)} \|\tilde{q}\|_{L^{1,2}(\Omega_\delta \times \Omega_\delta)} \right]$$