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A REGULARITY CRITERION FOR A 3D CHEMO-REPULSION SYSTEM AND ITS APPLICATION TO A BILINEAR OPTIMAL CONTROL PROBLEM*

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Abstract. In this paper, a bilinear optimal control problem associated to a 3D chemo-repulsion model with linear production is studied. The existence of weak solutions is proved, and a regularity criterion to get global-in-time strong solutions is established. As a consequence, the existence of a global optimal solution with bilinear control is deduced, and using a Lagrange multiplier theorem, first-order optimality conditions for local optimal solutions are derived.

Key words. chemo-repulsion and production model, weak solutions, strong solutions, bilinear optimal control, optimality conditions

AMS subject classifications. 35K51, 35Q92, 49J20, 49K20

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1. Introduction.

The chemotaxis phenomenon is understood as the directed movement of living organisms in response to chemical gradients. Keller and Segel [19] proposed a mathematical model that describes the chemotactic aggregation of cellular slime molds. These molds move preferentially toward relatively high concentrations of a chemical substance secreted by the amoebae themselves. Such a mechanism is called *chemo-attraction* with production. Conversely, when the regions of high chemical concentration generate a repulsive effect on the organisms, the phenomenon is called *chemo-repulsion*.

Bilinear control problems are a special class of nonlinear control problems in which a nonlinear term is constructed by multiplication of the control and state variables. In fact, the control acts as the coefficient of a reaction term depending linearly on the state. In this work we study an optimal control problem subject to a chemo-repulsion system with a linear production term and in which a bilinear control acts injecting or extracting a chemical substance on a subdomain of control $\Omega_c \subset \Omega$. Specifically, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$ and $(0,T)$ a time interval, with $0 < T < +\infty$. Then a control problem is studied which is related to the system in the time-space domain $Q := (0,T) \times \Omega$,

$$\left\{ \begin{array}{c} \partial_t u - \Delta u = \nabla \cdot (u \nabla v), \\ \partial_t v - \Delta v + v = u + f\, v\, 1_{\Omega_c}, \end{array} \right. \tag{1}$$

with initial conditions

$$u(0,\cdot) = u_0 \geq 0,\ v(0,\cdot) = v_0 \geq 0 \ \text{ in } \Omega, \tag{2}$$

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and nonflux boundary conditions

$$\frac{\partial u}{\partial \mathbf{n}} = 0, \quad \frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } (0,T) \times \partial\Omega, \tag{3}$$

where $\mathbf{n}$ denotes the outward unit normal vector to $\partial\Omega$. In (1), the unknowns are the cell density $u(t,x) \geq 0$ and chemical concentration $v(t,x) \geq 0$. The function $f = f(t,x)$ denotes a bilinear control acting in the chemical equation. It is observed that, in the subdomains of Ω_c where $f \geq 0$, such a control acts as a proliferation coefficient of the chemical substance, and conversely, where $f \leq 0$, the control acts as a degradation coefficient of the chemical substance. In particular, with this choice of the bilinear control, the solution (u,v) of system (1)–(3) always remains nonnegative. By considering a distributed control with a negative sign, the positivity of v could not be guaranteed.

System (1)–(3) without control (i.e., $f \equiv 0$) has been studied in [11] and [33]. In [11], the authors proved the global existence and uniqueness of smooth classical solutions in 2D domains and global existence of weak solutions in dimensions 3 and 4. In [33], on a bounded convex domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$, it is proved that a modified system of (1)–(3), changing the chemotactic term $\nabla \cdot (u\nabla v)$ by $\nabla \cdot (g(u)\nabla v)$ with an adequate density-dependent chemotactic function $g(u)$, has a unique global-in-time classical solution. This result is not applicable to problem (1) (even for $f = 0$) because $g(u) = u$ does not satisfy the hypothesis imposed in [33]. Both [11] and [33] (and many others studying chemotaxis models) are based in the abstract theory of classical solutions for quasi-linear parabolic systems (see, for instance, [3]) by applying the existence and uniqueness of local-in-time classical solutions and extensibility criteria of these classical solutions. In the case treated in this work, such a theory is not directly applicable due to the influence of the bilinear control term because no regularity is required for the derivatives of the control.

There is an extensive literature devoted to the study of control problems with PDEs; see, for instance, [2, 7, 8, 18, 20, 22, 25, 26, 32, 37] and references therein. In all previous works, the control is of distributed or boundary type. As far as we know, the literature related to optimal control problems with PDEs and bilinear control is scarce; see [5, 14, 17, 21, 36].

In the context of optimal control problems associated to chemotaxis models, the literature is also scarce. In [14, 30], a 1D problem is studied. In [14], the authors analyzed two problems for a chemoattractant model. The bilinear control acts on the whole Ω in the cells equation. The existence of optimal control is proved, and an optimality system is derived. Also, a numerical scheme for the optimality system is designed, and some numerical simulations are presented. In [30], a boundary control problem for a chemotaxis reaction-diffusion system is studied. The control acts on the boundary for the chemical substance, and the existence of optimal solution is proved. A distributed optimal control problem for a 2D model of cancer invasion has been studied in [12], proving the existence of an optimal solution and deriving an optimality system. Rodríguez-Bellido, Rueda-Gómez, and Villamizar-Roa [28] study a distributive optimal control problem related to a 3D stationary chemotaxis model coupled with the Navier–Stokes equations (*chemotaxis-fluid system*). The authors prove the existence of an optimal solution and derive an optimality system using a penalty method, taking into account that the relation control state is multivalued. Ryu and Yagi [29] studied an extreme problem for a chemoattractant 2D model in which the control variable is distributed in the chemical equation. They prove the existence of optimal solutions and derive an optimality system, using the fact that the state is differentiable with respect to the control. Other studies related to controllability for the nonstationary Keller–Segel model and nonstationary chemotaxis-fluid system can be consulted in [9] and [10], respectively.

In [17], an optimal bilinear control problem concerning strong solutions of system (1)–(3) in 2D domains was studied, proving the existence and uniqueness of global strong solutions and the existence of global optimal control. Moreover, using a Lagrange multiplier theorem, first-order optimality conditions are derived. Now, this paper can be seen as a 3D version of [17]. Similarly to [17], the main objective now is to prove the existence of global optimal solutions and to derive optimality conditions, which will be now more complicated because the PDE system is considered in 3D domains.

In fact, the regularity framework for system (1) in 2D and 3D problems is completely different. Note that, in 2D domains, the existence (and uniqueness) of a strong solution has been established in [17]. However, the strong regularity in 3D domains is not proved in general.

Therefore, in this case, two different types of solutions can be distinguished: *weak* and *strong*. The existence of weak solutions can be obtained under minimal assumptions (see Theorem 5). However, such a result is not sufficient to carry out the study of the control problem due to the lack of regularity of weak solutions. In order to overcome this problem, this paper introduces the regularity criterion $u \in L^{20/7}(Q)$ that allows obtaining a (unique) global-in-time strong solution of (1)–(3) (see Theorem 7). As far as it is known, there are no results of global-in-time regularity of weak solutions of system (1)–(3) in 3D domains. This is similar to what happens with the Navier–Stokes equations (see [34]).

Consequently, here we deal with strong solutions of (1)–(3), which allows analyzing the control problem. In fact, the existence of optimal control associated to strong solutions will be proved, assuming the existence of controls such that the associated strong solution of (1)–(3) exists. More concretely, it will be assumed that there exist weak solutions (u, v, f) of (1)–(3) satisfying the regularity criterion $u \in L^{20/7}(Q)$. As a matter of fact, this is true in the case of a control acting on the whole domain Ω, that is, when $\Omega_c = \Omega$, and strictly positive initial chemical concentration $v_0 \geq \alpha > 0$ in Ω; see Remark 6 below.

Following the ideas of [7, 8], it is considered the $L^{20/7}(Q)$-norm of the regularity criterion in the objective functional in such a way that any weak solution of (1)–(3) with a finite objective functional is also a strong solution.

This paper is organized as follows. In section 2, the notation has been fixed, the functional spaces to be used have been introduced, and a regularity result for linear parabolic-Neumann problems that will be used throughout this work is established. In section 3, the existence of weak solutions of system (1)–(3) is proved through a family of regularized problems (its existence is deduced in Appendix A). In section 4, a regularity criterion under which weak solutions of (1)–(3) are also strong solutions is established. Section 5 is dedicated to the study of a bilinear control problem related to strong solutions of system (1)–(3), proving the existence of an optimal solution and deriving the first-order optimality conditions based on a Lagrange multiplier argument in Banach spaces. Finally, a regularity result for these Lagrange multipliers is obtained.

2. Preliminaries. In this section, some notations will be introduced. The Lebesgue space $L^p(\Omega)$, $1 \leq p \leq +\infty$, with norm denoted by $\|\cdot\|_{L^p}$ will be used. In particular, the L^2-norm and its inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. The usual Sobolev spaces $W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \|\partial^\alpha u\|_{L^p} < +\infty,\ \forall|\alpha| \le m\}$, with norm denoted by $\|\cdot\|_{W^{m,p}}$, is considered. When $p = 2$, it is established that $H^m(\Omega) := W^{m,2}(\Omega)$, denoting the respective norm by $\|\cdot\|_{H^m}$. The space $W^{m,p}_{\mathbf{n}}(\Omega) = \{u \in W^{m,p}(\Omega) : \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\}$ $(m > 1 + 1/p)$, and its norm denoted by $\|\cdot\|_{W^{m,p}_{\mathbf{n}}}$ will also be used.

If X is a Banach space, $L^p(X)$ indicates the space of valued functions in X defined on the interval $[0, T]$ that are integrable in the Bochner sense, and its norm will be denoted by $\|\cdot\|_{L^p(X)}$. In particular, when X is a Lebesgue space $L^p(\Omega)$ or a Sobolev space $W^{m,q}(\Omega)$, $L^p(X)$ will be denoted simply by $L^p(L^q)$ or $L^p(W^{m,q})$, respectively. For simplicity, one defines $L^p(Q) := L^p(0, T; L^p(\Omega))$ if $p \neq +\infty$ and its norm by $\|\cdot\|_{L^p(Q)}$. In the case $p = +\infty$, $L^\infty(Q)$ means $L^\infty((0, T) \times \Omega)$, and its norm, is denoted by $\|\cdot\|_{L^\infty(Q)}$. It is denoted by $C([0, T]; X)$ the space of continuous functions from $[0, T]$ into a Banach space X, whose norm is given by $\|\cdot\|_{C(X)}$. The topological dual space of a Banach space X will be denoted by X' and the duality for a pair X and X' by $\langle\cdot,\cdot\rangle_{X'}$ or simply by $\langle\cdot,\cdot\rangle$ unless this leads to ambiguity. Moreover, the letters C, K, C_0, K_0, C_1, $K_1, \ldots,$ denote positive constants, independent of the state (u, v) and control f, but its value may change from line to line.

In order to study the existence of solution of system (1)–(3), the space is defined by

$$\widehat{W}^{2-2/p,p}(\Omega) := \begin{cases} W^{2-2/p,p}(\Omega) & \text{if} \quad p < 3, \\ W^{2-2/p,p}_{\mathbf{n}}(\Omega) & \text{if} \quad p > 3. \end{cases}$$

The following regularity result for the heat equation we will often be used (see [13, p. 344]).

Theorem 1. *For $\Omega \in C^2$, let $1 < p < +\infty$ $(p \neq 3)$, $u_0 \in \widehat{W}^{2-2/p,p}(\Omega)$, and $g \in L^p(Q)$. Then the problem*

$$\begin{cases} \partial_t u - \Delta u = g & \text{in } Q, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } (0,T) \times \partial\Omega \end{cases}$$

admits a unique solution u such that

$$u \in C([0,T]; \widehat{W}^{2-2/p,p}) \cap L^p(W^{2,p}_{\mathbf{n}}), \quad \partial_t u \in L^p(Q).$$

Moreover, there exists a positive constant $C := C(p, \Omega, T)$ such that

$$\|u\|_{C(\widehat{W}^{2-2/p,p})} + \|\partial_t u\|_{L^p(Q)} + \|u\|_{L^p(W^{2,p})} \le C(\|g\|_{L^p(Q)} + \|u_0\|_{\widehat{W}^{2-2/p,p}}).$$

For simplicity, in what follows, the following notation for specific Banach spaces will be used frequently in the paper:

$$X_p := \{u \in C([0,T]; \widehat{W}^{2-2/p,p}) \cap L^p(W^{2,p}_{\mathbf{n}}) : \partial_t u \in L^p(Q)\} \quad (p > 1),$$

$$X := C([0,T]; L^2) \cap L^2(H^1). \tag{4}$$

Moreover, its norm will be denoted by $\|\cdot\|_{X_p}$ and $\|\cdot\|_X$, respectively.

The Leray–Schauder fixed-point theorem will be used several times to prove the existence of solution for some different problems.

THEOREM 2 (Leray–Schauder fixed-point theorem). *Let $\mathcal{X}$ be a Banach space and $T : \mathcal{X} \to \mathcal{X}$ a continuous and compact operator. If the set*

$$\{x \in \mathcal{X} : x = \alpha T x \quad \text{for some } 0 \le \alpha \le 1\}$$

is bounded, then T has (at least) a fixed point.

In this paper, the Leray–Schauder theorem will be applied considering the following steps.

LEMMA 1. *$T : \mathcal{X} \to \mathcal{Y}$ is well defined and maps bounded sets of $\mathcal{X}$ into bounded sets of $\mathcal{Y}$.*

LEMMA 2. *$\mathcal{Y}$ is compactly embedded into $\mathcal{X}$.*

LEMMA 3. *$T : \mathcal{X} \to \mathcal{X}$ is a continuous and compact operator.*

LEMMA 4. *The set $\{x \in \mathcal{X} : x = \alpha T x \quad \text{for some } 0 \le \alpha \le 1\}$ is bounded in $\mathcal{X}$ (with respect to α).*

In this paper, the following two compactness results will be applied.

THEOREM 3 (Aubin–Lions lemma). *Let $\mathcal{X}$, B, and $\mathcal{Y}$ be reflexive Banach spaces such that $\mathcal{X} \subset B \subset \mathcal{Y}$, with compact embedding $\mathcal{X} \mapsto B$ and continuous embedding $B \hookrightarrow \mathcal{Y}$. It is defined by*

$$W = \{w : w \in L^{p_0}(0,T;\mathcal{X})),\ \partial_t w \in L^{p_1}(0,T;\mathcal{Y})\}$$

for a finite $T > 0$ and p_0, $p_1 \in (1, +\infty)$. Then the injection of W into $L^{p_0}(0,T;B)$ is compact.

Proof. See [23, Theorem 5.1, p. 58]. ☐

THEOREM 4 (Simon's compactness result). *Let $\mathcal{X}$, B, and $\mathcal{Y}$ be Banach spaces such that $\mathcal{X} \subset B \subset \mathcal{Y}$, with compact embedding $\mathcal{X} \mapsto B$ and continuous embedding $B \hookrightarrow \mathcal{Y}$. Let F be a bounded set in $L^\infty(0,T;\mathcal{X})$ such that the set $\partial_t F = \{\frac{\partial f}{\partial t};\ f \in F\}$ is bounded in $L^r(0,T;\mathcal{Y})$ for some $r > 1$. Then F is relatively compact in $C([0,T];B)$.*

Proof. See [31, Corollary 4]. ☐

Throughout this paper, the following equivalent norms in $H^1(\Omega)$ and $H^2(\Omega)$ will be used, respectively (see [27] for details):

$$\|u\|_{H^1}^2 \simeq \|\nabla u\|^2 + \left(\int_\Omega u\right)^2 \quad \forall\, u \in H^1(\Omega), \tag{5}$$

$$\|u\|_{H^2}^2 \simeq \|\Delta u\|^2 + \left(\int_\Omega u\right)^2 \quad \forall\, u \in H^2_{\mathbf{n}}(\Omega). \tag{6}$$

Also, the classical interpolation inequality in 3D domains will be used:

$$\|u\|_{L^3} \le C\|u\|^{1/2}\|u\|_{H^1}^{1/2} \quad \forall\, u \in H^1(\Omega). \tag{7}$$

To obtain the regularity of different terms in the equations, some embedding results will be used. In the following, some results in this setting are established.

As a consequence of the interpolation inequality

$$\|u\|_{L^p} \le \|u\|_{L^{p_1}}^{1-\theta}\|u\|_{L^{p_2}}^{\theta}, \quad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2} \quad \text{and} \quad \theta \in [0,1],$$

the following $L^p(L^q)$-interpolation result can be deduced.

Lemma 5. *Let* $p_1, p_2, q_1, q_2, p, q \geq 1$ *be such that*

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2} \quad \textit{and} \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \ \textit{with } \theta \in [0,1].$$

Then

$$L^{p_1}(L^{q_1}) \cap L^{p_2}(L^{q_2}) \hookrightarrow L^p(L^q). \tag{8}$$

As a consequence of the Sobolev embeddings [1, Theorem 7.58, p. 218],

$$W^{r,p}(\Omega) \hookrightarrow H^s(\Omega), \quad r, s > 0,\ 1 < p < 2, \text{ with } s = N\left(\frac{1}{2} - \frac{1}{p}\right) + r, \tag{9}$$

$$W^{r,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{with } \frac{1}{q} = \frac{1}{p} - \frac{r}{N}, \tag{10}$$

where N is the space dimension. The following $L^p(H^s)$-interpolation result can be obtained.

Lemma 6 ([24, Theorem 9.6, p. 49]). *Let* $p_1, p_2, p \geq 1$ *and* $s_1, s_2, s \geq 0$ *be such that*

$$s = (1-\theta)s_1 + \theta s_2 \quad \textit{and} \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \ \textit{with } \theta \in [0,1].$$

Then $L^{p_1}(H^{s_1}) \cap L^{p_2}(H^{s_2}) \hookrightarrow L^p(H^s)$. *In particular, one has* $L^{p_1}(H^{s_1}) \cap L^{p_2}(H^{s_2}) \hookrightarrow L^p(L^q)$ *with* $\frac{1}{q} = \frac{1}{2} - \frac{s}{N}$.

Corollary 1. $L^\infty(L^2) \cap L^2(H^1) \hookrightarrow L^{10/3}(Q)$ *and* $L^\infty(H^1) \cap L^2(H^2) \hookrightarrow L^{10}(Q)$.

Using the Sobolev embedding (10) and the Gagliardo–Nirenberg inequality (see [15, Theorem 10.1]),

$$W^{s,p_1}(\Omega) \cap L^{p_2}(\Omega) \hookrightarrow L^p(\Omega), \quad \text{with } \frac{1}{p} = \theta\left(\frac{1}{p_1} - \frac{s}{N}\right) + \frac{1-\theta}{p_2} \text{ and } \theta \in [0,1],$$

the following Bochner–Sobolev-L^p embedding result can be deduced.

Lemma 7. *Let* $p_1, q_1, p_2, p, q \geq 1$ *be such that*

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \theta\left(\frac{1}{p_1} - \frac{r}{N}\right) \ \textit{and} \ \frac{1}{p} = \frac{\theta}{p_2}, \ \textit{with } \theta \in [0,1] \textit{ and } r > 0.$$

Then $L^\infty(L^{q_1}) \cap L^{p_2}(W^{r,p_1}) \hookrightarrow L^p(L^q)$.

Lemma 8 ([13]). *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain. Then the interpolation inequality*

$$\|v\|_{W^{\alpha,r}(\Omega)} \leq C\, \|v\|^{\lambda}_{W^{\beta,p}(\Omega)} \|v\|^{1-\lambda}_{W^{\gamma,q}(\Omega)}, \quad 0 \leq \lambda \leq 1, \tag{11}$$

holds for $0 \leq \alpha,\ \beta,\ \gamma \leq 1$, $1 < p,\ q,\ r < \infty$, $\alpha = \lambda\,\beta + (1-\lambda)\,\gamma$, $\frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}$.

When using the Leray–Schauder fixed-point theorem, Lemma 2 needs to be proved. The following results will be of great help for it.

Lemma 9. *The space* $X_{20/13}$ *is compactly embedded in* X.

Proof. Note that if $w \in X_{20/13}$, then using (9) and Lemma 6,

$$w \in L^\infty(W^{7/10,20/13}) \cap L^{20/13}(W^{2,20/13}) \hookrightarrow L^\infty(H^{1/4}) \cap L^{20/13}(H^{31/20}) \hookrightarrow L^p(H^{1/4+2/p}) \tag{12}$$

with $p \geq 20/13$. Thus, again from the definition of $X_{20/13}$ and (12),

$$w \in L^\infty(H^{1/4}) \cap L^2(H^{5/4}), \quad \partial_t w \in L^{20/13}(Q).$$

Therefore, the Aubin–Lions lemma (Theorem 3) and Simon's compactness result (Theorem 4) can be applied, allowing us to deduce that $X_{20/13}$ is compactly embedded in X. ∎

Since $X_p \subset X_{20/13}$ for any $p \geq 20/13$, the following result can be concluded.

COROLLARY 2. *The space X_p is compactly embedded in X for any $p \geq 20/13$.*

3. Existence of weak solutions of problem (1)–(3).

DEFINITION 1 (weak solution (u, v)). *Let $f \in L^4(Q_c) := L^4(0, T; L^4(\Omega_c))$, $u_0 \in L^{p_0}(\Omega)$ for some $p_0 > 1$, $v_0 \in H^1(\Omega)$ with $u_0 \geq 0$, and $v_0 \geq 0$ in Ω. A pair (u, v) is called a weak solution of problem (1)–(3) in $(0, T)$ if $u \geq 0$, $v \geq 0$ a.e. in Q:*

$$u \in L^{5/3}(Q) \cap L^{5/4}(W^{1,5/4}), \ \partial_t u \in L^{10/9}((W^{1,10})'), \tag{13}$$

$$v \in L^\infty(H^1) \cap L^2(H^2), \ \partial_t v \in L^{5/3}(Q). \tag{14}$$

The following variational formulation holds for the u-equation:

$$\int_0^T \langle \partial_t u, \overline{u} \rangle + \int_0^T (\nabla u, \nabla \overline{u}) + \int_0^T (u \nabla v, \nabla \overline{u}) = 0 \quad \forall \overline{u} \in L^{10}(W^{1,10}). \tag{15}$$

The v-equation $(1)_2$ holds pointwisely a.e. $(t, x) \in Q$, and the boundary conditions for v $(3)_2$ and the initial conditions for (u, v) (2) are satisfied.

Remark 1. Taking $\overline{u} = 1$ in (15), one has

$$u \in L^\infty(L^1) \ \text{ and } \ \int_\Omega u(t) = \int_\Omega u_0 := m_0.$$

In addition to this conservation property, the cornerstone to prove the existence of a weak solution of (1)–(3) is the following energy law, which can be computed only in a formal way, testing $(1)_1$ by $\ln(u)$ plus $(1)_2$ by $-\Delta v$. Since chemotactic and production terms cancel, one arrives at

$$\frac{d}{dt} E(u(t), v(t)) + \int_\Omega \left(4|\nabla \sqrt{u}|^2 + |\Delta v|^2 + |\nabla v|^2 \right) = -\int_\Omega f\, v\, (\Delta v)\, 1_{\Omega_c},$$

where $E(u, v) := \int_\Omega (u \ln(u) - u + \frac{1}{2}|\nabla v|^2)$. See (139) in Appendix A, to find a rigorous proof of an energy law for a regularized problem.

Finally, each term of (15) has a sense. In particular, from (13)–(14) one has that $u \nabla v \in L^{10/9}(Q)$, and therefore $\int_0^T (u \nabla v, \nabla \overline{u})$ is finite for all $\overline{u} \in L^{10}(W^{1,10})$. Indeed, from Corollary 1 it is known that $\nabla v \in L^\infty(L^2) \cap L^2(H^1) \hookrightarrow L^{10/3}(Q)$, which, jointly with the fact that $u \in L^{5/3}(Q)$, implies that $u \nabla v \in L^{10/9}(Q)$.

THEOREM 5 (existence of weak solutions of (1)–(3)). *There exists a weak solution (u, v) of system (1)–(3) in the sense of Definition 1.*

The proof of this theorem follows from the two next subsections.

3.1. Regularized problem. In order to prove Theorem 5, the following family of regularized problems related to system (1)–(3) will be studied for any $\varepsilon \in (0,1)$. Given an adequate regularization $(u_0^\varepsilon, v_0^\varepsilon)$ of initial data (u_0, v_0), it is defined by $(u^\varepsilon, z^\varepsilon)$ as the solution of

$$\left\{\begin{array}{c} \partial_t u^\varepsilon - \Delta u^\varepsilon = \nabla \cdot (u^\varepsilon \nabla v(z^\varepsilon)) \quad \text{in } Q, \\ \partial_t z^\varepsilon - \Delta z^\varepsilon + z^\varepsilon = u^\varepsilon + f\, v(z^\varepsilon)_+ 1_{\Omega_c} \quad \text{in } Q, \\ u^\varepsilon(0) = u_0^\varepsilon,\ z^\varepsilon(0) = v_0^\varepsilon - \varepsilon \Delta v_0^\varepsilon \quad \text{in } \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial \mathbf{n}} = 0,\ \dfrac{\partial z^\varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on } (0,T) \times \partial\Omega, \end{array}\right. \tag{16}$$

where $v^\varepsilon(t,\cdot) := v(z^\varepsilon(t,\cdot))$ for any $t \in [0,T]$ is the unique solution of the problem

$$\left\{\begin{array}{l} v^\varepsilon - \varepsilon \Delta v^\varepsilon = z^\varepsilon \quad \text{in } \Omega, \\ \dfrac{\partial v^\varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \end{array}\right. \tag{17}$$

and $v_+ := \max\{v, 0\} \geq 0$.

The initial conditions u_0^ε and v_0^ε such that $u_0^\varepsilon \geq 0$ a.e. in Ω, $(u_0^\varepsilon, v_0^\varepsilon - \varepsilon \Delta v_0^\varepsilon) \in W^{4/5,5/3}(\Omega) \times W_{\mathbf{n}}^{7/5,10/3}(\Omega)$, are chosen, and

$$(u_0^\varepsilon, v_0^\varepsilon) \to (u_0, v_0) \quad \text{in } L^{p_0}(\Omega) \times H^1(\Omega) \text{ as } \varepsilon \to 0 \quad \text{for } p_0 > 1. \tag{18}$$

In particular,

$$z^\varepsilon(0) = v_0^\varepsilon - \varepsilon \Delta v_0^\varepsilon \to v_0 \quad \text{in } (H^1(\Omega))' \text{ as } \varepsilon \to 0. \tag{19}$$

In the remainder of this section, $v(z^\varepsilon)$ will be denoted only by v^ε.

DEFINITION 2 (strong solution $(u^\varepsilon, z^\varepsilon)$). *Let $u_0^\varepsilon \in W^{4/5,5/3}(\Omega)$, $v_0^\varepsilon - \varepsilon \Delta v_0^\varepsilon \in W_{\mathbf{n}}^{7/5,10/3}(\Omega)$ with $u_0^\varepsilon \geq 0$ in Ω, and $f \in L^4(Q_c)$. It is said that a pair $(u^\varepsilon, z^\varepsilon)$ is a strong solution of problem* (16) *in $(0,T)$ if $u^\varepsilon \geq 0$ a.e. in Q:*

$$(u^\varepsilon, z^\varepsilon) \in X_{5/3} \times X_{10/3}.$$

Equations $(16)_1$–$(16)_2$ *hold a.e. $(t,x) \in Q$, and the initial and boundary conditions* $(16)_3$–$(16)_4$ *are satisfied.*

Remark 2. Notice that each function $w \in X_p$ in particular belongs to $C([0,T]; \widehat{W}^{2-2/p,p})$. Since the regularity obtained is $(u^\varepsilon, z^\varepsilon) \in X_{5/3} \times X_{10/3}$, from Theorem 1 it is necessary to impose $u_0^\varepsilon \in \widehat{W}^{2-2/(5/3),5/3}(\Omega) = W^{4/5,5/3}(\Omega)$ (in this case, $p = 5/3 < 3$) and $z_0^\varepsilon \in \widehat{W}^{2-2/(10/3),10/3}(\Omega) = W_{\mathbf{n}}^{7/5,10/3}(\Omega)$ (in this case, $p = 10/3 > 3$). Additionally, assuming more regularity of data $(u_0^\varepsilon, z_0^\varepsilon)$, more regularity can be deduced from the solution $(u^\varepsilon, z^\varepsilon)$ making a bootstrapping argument using Theorem 1, as in the proof of Theorem 7 below. For instance, if $(u_0^\varepsilon, z_0^\varepsilon) \in W_{\mathbf{n}}^{3/2,4}(\Omega)^2$, then $(u^\varepsilon, z^\varepsilon) \in X_4 \times X_4$.

THEOREM 6. *There exists a strong solution $(u^\varepsilon, z^\varepsilon) \in X_{5/3} \times X_{10/3}$ of system* (16) *in $(0,T)$ in the sense of Definition* 2.

The proof of Theorem 6 is carried out in Appendix A.

3.2. Proof of Theorem 5. Taking limits as $\varepsilon \to 0$. Following the proof of Lemma 15 in Appendix A, using in particular the conservation property (138) and the energy inequality (143), and taking into account that $\int_\Omega (u_0^\varepsilon + 1)\ln(u_0^\varepsilon + 1) \leq C\|u_0^\varepsilon + 1\|_{L^{p_0}}$ for any $p_0 > 1$ and (18), the following estimates can be deduced from the Gronwall lemma (uniformly with respect to ε):

(20)
$$\begin{cases} \{\nabla\sqrt{u^\varepsilon + 1}\}_{\varepsilon>0} \quad \text{is bounded in } L^2(Q), \\ \{\sqrt{u^\varepsilon + 1}\}_{\varepsilon>0} \quad \text{is bounded in } L^\infty(L^2) \cap L^2(L^6) \hookrightarrow L^{10/3}(Q) \cap L^8(L^{12/5}), \\ \{v^\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } L^\infty(H^1) \cap L^2(H^2) \hookrightarrow L^{10}(Q), \\ \{\sqrt{\varepsilon}\Delta v^\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } L^\infty(L^2) \cap L^2(H^1). \end{cases}$$

From $(20)_2$, one has

$$\{u^\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } L^\infty(L^1) \cap L^{5/3}(Q) \cap L^4(L^{6/5}). \tag{21}$$

Moreover, taking into account that $\nabla u^\varepsilon = 2\sqrt{u^\varepsilon + 1}\nabla\sqrt{u^\varepsilon + 1}$, from $(20)_1$ and $(20)_2$ it is deduced that

$$\{u^\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } L^{5/4}(W^{1,5/4}). \tag{22}$$

From $(20)_3$, one also has that $\{\nabla v^\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(L^2)\cap L^2(H^1) \hookrightarrow L^{10/3}(Q)$, which, jointly with (21), implies that

$$\{u^\varepsilon \nabla v^\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } L^{10/9}(Q). \tag{23}$$

Looking at the u^ε-equation and previous estimates (22) and (23), it is deduced that

$$\{\partial_t u^\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } L^{10/9}((W^{1,10})'). \tag{24}$$

From $(20)_{3,4}$ and the equality $z^\varepsilon = v^\varepsilon - \varepsilon\Delta v^\varepsilon$, one has that

$$\{z^\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } L^\infty(L^2) \cap L^2(H^1). \tag{25}$$

Looking at the z^ε-equation and previous estimates $(20)_3$, (21), and (25), it is deduced that

$$\{\partial_t z^\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } L^{5/3}((H^1)'). \tag{26}$$

Notice that from (17) and $(20)_4$, it is obtained that

$$z^\varepsilon - v^\varepsilon = -\varepsilon\Delta v^\varepsilon \to 0 \quad \text{as } \varepsilon \to 0 \quad \text{in the } L^\infty(L^2) \cap L^2(H^1)\text{-norm}. \tag{27}$$

Therefore, from $(20)_3$, (21), (22), and (27), it is deduced that that there exist limit functions (u, v) such that

$$\begin{cases} u \in L^{5/3}(Q) \cap L^{5/4}(W^{1,5/4}), \\ v \in L^\infty(H^1) \cap L^2(H^2), \end{cases}$$

and for some subsequence of $\{(u^\varepsilon, v^\varepsilon, z^\varepsilon)\}_{\varepsilon>0}$, still denoted by $\{(u^\varepsilon, v^\varepsilon, z^\varepsilon)\}_{\varepsilon>0}$, the following convergences hold as $\varepsilon \to 0$:

$$\begin{cases} u^\varepsilon \to u \quad \text{weakly in } L^{5/3}(Q) \cap L^{5/4}(W^{1,5/4}), \\ v^\varepsilon \to v \quad \text{weakly in } L^2(H^2) \text{ and weakly* in } L^\infty(H^1), \\ z^\varepsilon \to v \quad \text{weakly in } L^2(H^1) \text{ and weakly* in } L^\infty(L^2), \\ \partial_t u^\varepsilon \to \partial_t u \quad \text{weakly* in } L^{10/9}((W^{1,10})'), \\ \partial_t z^\varepsilon \to \partial_t v \quad \text{weakly* in } L^{5/3}((H^1)'). \end{cases} \tag{28}$$

It will be verified that (u,v) is a weak solution of (1)–(3). Indeed, from (21), (22), and (24), by applying compactness results Theorems 3 and 4, it is deduced that

$$\{u^\varepsilon\}_{\varepsilon>0} \text{ is relatively compact in } L^{5/4}(L^2)\cap C([0,T];(W^{1,q})')\ (q>3). \tag{29}$$

In particular, since u^ε is bounded in $L^{5/3}(Q)$,

$$\{u^\varepsilon\}_{\varepsilon>0} \text{ is relatively compact in } L^p(Q) \quad \forall p<5/3. \tag{30}$$

Taking into account (23), the weak convergence

$$u^\varepsilon\nabla v^\varepsilon \to \chi \quad \text{weakly in } L^{10/9}(Q) \tag{31}$$

is obtained. From $(28)_2$, one also has $\nabla v^\varepsilon \to \nabla v$ weakly in $L^{10/3}(Q)$. Thus, (30) and (31) allow us to identify χ with $u\nabla v$, and

$$u^\varepsilon\nabla v^\varepsilon \to u\nabla v \quad \text{weakly in } L^{10/9}(Q). \tag{32}$$

Furthermore, from $(28)_3$ and $(28)_5$ and again using the compactness results from Theorems 3 and 4, it is obtained that

$$z^\varepsilon \to v \ \text{ strongly in } \ L^2(Q)\cap C([0,T];(H^1)'). \tag{33}$$

In particular, from (29) and (33), $(u^\varepsilon(0),z^\varepsilon(0))$ converges to $(u(0),v(0))$ in $(W^{1,q})' \times (H^1)'$; then from (18), (19), and the uniqueness of the limit, the identification $(u(0),v(0))=(u_0,v_0)$ holds in $L^{p_0}(\Omega)\times H^1(\Omega)$ $(p_0>1)$, which is the initial condition given in (2).

From (27), $(28)_2$, and (33), it is deduced that v^ε converges to v strongly in $L^2(Q)$, which implies that

$$v_+^\varepsilon \to v_+ \ \text{ strongly in } L^2(Q).$$

Then, using $(20)_3$ and $f\in L^4(Q_c)$, it is deduced that

$$f\,v_+^\varepsilon 1_{\Omega_c} \to f\,v_+ 1_{\Omega_c} \quad \text{weakly in } L^{20/7}(Q). \tag{34}$$

Therefore, taking to the limit in the regularized problem (16), as $\varepsilon\to 0$, and taking into account (28), (32), and (34), it is concluded that the limit (u,v) satisfies the weak formulation

$$\int_0^T \langle \partial_t u,\overline{u}\rangle + \int_0^T (\nabla u,\nabla\overline{u}) + \int_0^T (u\nabla v,\nabla\overline{u}) = 0 \quad \forall \overline{u}\in L^{10}(W^{1,10}), \tag{35}$$

$$\int_0^T \langle \partial_t v,\overline{z}\rangle + \int_0^T (\nabla v,\nabla\overline{z}) + \int_0^T (v,\bar{z}) = \int_0^T (u,\overline{z}) + \int_0^T (f\,v_+1_{\Omega_c},\overline{z}) \ \ \forall \overline{z}\in L^{5/2}(H^1). \tag{36}$$

Note that (35) is exactly the variational formulation given in (15). Moreover, integrating by parts in (36) and using that $u\in L^{5/3}(Q)$ and $v\in L^2(H^2_{\mathbf{n}})$, it can be deduced that

$$\partial_t v - \Delta v + v = u + f\,v_+ 1_{\Omega_c} \quad \text{in } L^{5/3}(Q). \tag{37}$$

Finally, the positivity of (u,v) will be checked. Indeed, the positivity of u follows from (29) and the fact that $u^\varepsilon \geq 0$ a.e. $(t,x)\in Q$ (see Lemma 15 in Appendix A). In order to check that $v \geq 0$, (37) is tested by $v_- := \min\{v, 0\} \leq 0$. Taking into account that $u \geq 0$ and using that $v_- = 0$ if $v \geq 0$, $\nabla v_- = \nabla v$ if $v \leq 0$, and $\nabla v_- = 0$ if $v > 0$, it is obtained that

$$\frac{1}{2}\frac{d}{dt}\|v_-\|^2 + \|\nabla v_-\|^2 + \|v_-\|^2 = (u, v_-) + (f\, v_+ 1_{\Omega_c}, v_-) \leq 0,$$

which implies that $v_- \equiv 0$; then $v \geq 0$ a.e. $(t, x) \in Q$ (the fact of taking v_+ in the control term is used here to guarantee the positivity of v). From (37) and $v_+ \equiv v$, v satisfies the v-equation $(1)_2$ pointwisely a.e. $(t, x) \in Q$.

4. Regularity criterion. In this section, a regularity criterion of system (1)–(3) will be given.

DEFINITION 3 (strong solution (u, v)). *Let $f \in L^4(Q_c)$ and $(u_0, v_0) \in W_{\mathbf{n}}^{3/2,4}(\Omega)^2$ with $u_0 \geq 0$ and $v_0 \geq 0$ a.e. in Ω. A pair (u, v) is called a strong solution of problem* (1)–(3) *in $(0, T)$ if $u \geq 0$, $v \geq 0$ a.e. in Q,*

$$(u, v) \in X_4 \times X_4, \tag{38}$$

the system (1) *holds a.e. $(t, x) \in Q$, and the initial and boundary conditions* (2) *and* (3) *are satisfied.*

Remark 3. Using the interpolation inequality (7) and the Gronwall lemma and following similar arguments to those presented in [17] (in the 2D case), the uniqueness of strong solutions of (1)–(3) can be deduced. In fact, comparing two possible solutions (u_i, v_i), for $i = 1, 2$, with $(u, v) = (u_1 - u_2, v_1 - v_2)$ and testing by u and $-\Delta v$, one has uniqueness via the Gronwall lemma if $u_1, \nabla v_2 \in L^4(0, T; L^6)$.

THEOREM 7 (regularity criterion). *Let (u, v) be a weak solution of* (1)–(3)*. If, in addition, $(u_0, v_0) \in W_{\mathbf{n}}^{3/2,4}(\Omega)^2$ and the regularity criterion*

$$u \in L^{20/7}(Q) \tag{39}$$

holds, then (u, v) is a strong solution of (1)–(3) *in the sense of Definition* 3*. Moreover, there exists a positive constant $K = K(\|u_0\|_{W_{\mathbf{n}}^{3/2,4}}, \|v_0\|_{W_{\mathbf{n}}^{3/2,4}}, \|f\|_{L^4(Q)})$ such that*

$$\|(u, v)\|_{X_4 \times X_4} \leq K. \tag{40}$$

Remark 4. From the definition of X_4 and some Sobolev continuous embeddings, one has

$$w \in X_4 \hookrightarrow L^\infty(W^{3/2,4}) \hookrightarrow L^\infty(Q) \quad \text{and} \quad \Delta w \in L^4(Q). \tag{41}$$

Moreover, looking at the proof of Step 5 of Theorem 7, from (70) it is deduced in particular that

$$w \in X_4 \quad \Rightarrow \quad \nabla w \in L^{20}(Q). \tag{42}$$

The proof of Theorem 7 follows from the next subsection.

4.1. Proof of Theorem 7. Starting from the weak regularity of (u, v) and the hypothesis $u \in L^{20/7}(Q)$, the regularity for u and v will be improved several times using a bootstrapping argument, arriving at the optimal regularity $(u, v) \in X_4 \times X_4$.

The proof of Theorem 7 is carried out in four steps:

Step 1: $v \in X_{20/7}$ and $\nabla v \in L^{20/3}(Q)$.

From Theorem 5, it is known that there exists a weak solution (u, v) of system (1)–(3) in the sense of Definition 1. Thus, since $v \in L^\infty(H^1) \cap L^2(H^2)$, $v \in L^{10}(Q)$ (Corollary 1), then $f\, v\, 1_{\Omega_c} \in L^{20/7}(Q)$, which implies, using hypothesis (39), that $u + f\, v\, 1_{\Omega_c} \in L^{20/7}(Q)$. Then, applying Theorem 1 (for $p = 20/7$) to $(1)_2$, one has $v \in X_{20/7}$. In particular, using Sobolev embeddings, it holds that

$$v \in L^\infty(Q), \tag{43}$$

$$\nabla v \in L^\infty(L^4) \cap L^{20/7}(W^{1,20/7}) \hookrightarrow L^\infty(L^4) \cap L^{20/7}(L^{60}). \tag{44}$$

From (44) and applying Lemma 5 (for $p_1 = \infty$, $q_1 = 4$, $p_2 = 20/7$, and $q_2 = 60$, hence $p = q = 20/3$), one has

$$\nabla v \in L^{20/3}(Q). \tag{45}$$

Step 2: $u \in L^\infty(L^2) \cap L^2(H^1)$.

Starting from $u \in L^{20/7}(Q) \cap L^{5/4}(W^{1,5/4})$ and $v \in X_{20/7}$, the regularity of u is improved by a bootstrapping argument in eight substeps:

(i) $u \in X_{20/19}$. Using that $(u, \Delta v) \in L^{20/7}(Q) \times L^{20/7}(Q)$, $u\Delta v \in L^{10/7}(Q)$, and using that $(\nabla u, \nabla v) \in L^{5/4}(Q) \times L^{20/3}(Q)$, $\nabla u \cdot \nabla v \in L^{20/19}(Q)$. Therefore,

$$\nabla \cdot (u\nabla v) = u\Delta v + \nabla u \cdot \nabla v \in L^{20/19}(Q).$$

Thus, applying Theorem 1 (for $p = 20/19$) to $(1)_1$, it is obtained that $u \in X_{20/19}$.

(ii) $u \in X_{10/9}$. Since $u \in X_{20/19}$,

$$u \in L^\infty(W^{1/10,20/19}) \cap L^{20/19}(W^{2,20/19}).$$

Observe that, denoting by $D^{1/10}u$ the 1/10-derivatives of u,

$$D^{1/10}u \in L^\infty(L^{20/19}) \cap L^{20/19}(W^{19/10,20/19}) \hookrightarrow L^\infty(L^{20/19}) \cap L^{20/19}(W^{1,20/13}). \tag{46}$$

Applying Lemma 8 to (46) for $(\alpha, r) = (9/10, r)$, $(\beta, p) = (1, 20/13)$, and $(\gamma, q) = (0, 20/19)$, $(\lambda, r) = (9/10, 25/17)$ is obtained, and $D^{1/10}u$ satisfies

$$\|D^{1/10}u\|^s_{W^{9/10,25/17}} \le C\|D^{1/10}u\|^{s\,\lambda}_{W^{1,20/13}} \|D^{1/10}u\|^{s(1-\lambda)}_{L^{20/19}}, \tag{47}$$

where expression (47) is integrable in time if $s\,\lambda = 20/19$, and thus $s = 200/171$. Therefore, $u \in L^{200/171}(W^{1,25/17})$ and

$$\nabla u \in L^{200/171}(L^{25/17}). \tag{48}$$

Moreover, from (44), using (8) (for $p_1 = \infty$, $q_1 = 4$, $p_2 = 20/7$, $q_2 = 60$, and $p = 10$, hence $q = 60/11$), it is obtained that

$$\nabla v \in L^\infty(L^4) \cap L^{10}(L^{60/11}). \tag{49}$$

Thus, from (48) and (49), $\nabla u \cdot \nabla v \in L^{200/171}(L^{100/93}) \cap L^{200/191}(L^{300/259})$ is deduced. Then, owing to (8) applied to $(p_1, q_1) = (200/171, 100/93)$ and $(p_2, q_2) = (200/191, 300/259)$ implies that $p = q = 10/9$; hence,

$$\nabla u \cdot \nabla v \in L^{10/9}(Q).$$

Since $u\Delta v \in L^{10/7}(Q)$, $\nabla\cdot(u\nabla v) \in L^{10/9}(Q)$ holds. Then, applying Theorem 1 (for $p = 10/9$) to $(1)_1$, it is deduced that $u \in X_{10/9}$.

(iii) $u \in X_{20/17}$. Since $u \in X_{10/9}$,

$$u \in L^\infty(W^{1/5,10/9}) \cap L^{10/9}(W^{2,10/9}).$$

Observe that, denoting by $D^{1/5}u$ the 1/5-derivatives of u,

$$D^{1/5}u \in L^\infty(L^{10/9}) \cap L^{10/9}(W^{9/5,10/9}) \hookrightarrow L^\infty(L^{10/9}) \cap L^{10/9}(W^{1,30/19}). \tag{50}$$

Applying Lemma 8 to (50) for $(\alpha, r) = (4/5, r)$, $(\beta, p) = (1, 30/19)$, and $(\gamma, q) = (0, 10/9)$, we obtain $(\lambda, r) = (4/5, 150/103)$, and $D^{1/5}u$ satisfies

$$\|D^{1/5}u\|^s_{W^{4/5,150/103}} \le C\|D^{1/5}u\|^{s\,\lambda}_{W^{1,30/19}}\|D^{1/5}u\|^{s(1-\lambda)}_{L^{10/9}}, \tag{51}$$

where expression (51) is integrable in time if $s\,\lambda = 10/9$, and thus $s = 25/18$. Therefore, $u \in L^{25/18}(W^{1,150/103})$ and

$$\nabla u \in L^{25/18}(L^{150/103}). \tag{52}$$

Now, from (44), using (8) (for $p_1 = \infty$, $q_1 = 4$, $p_2 = 20/7$, $q_2 = 60$, and $p = 5$, hence $q = 60/7$),

$$\nabla v \in L^\infty(L^4) \cap L^5(L^{60/7}),$$

which jointly with (52) implies that $\nabla u \cdot \nabla v \in L^{25/18}(L^{300/281}) \cap L^{25/23}(L^{300/241})$. Then using (8) with $(p_1, q_1) = (25/18, 300/281)$, $(p_2, q_2) = (25/23, 300/241)$ implies that $p = q = 20/17$; hence,

$$\nabla u \cdot \nabla v \in L^{20/17}(Q).$$

Since $u\Delta v \in L^{10/7}(Q)$, it is deduced that $\nabla\cdot(u\nabla v) \in L^{20/17}(Q)$. Then, applying Theorem 1 (for $p = 20/17$) to $(1)_1$, one has that $u \in X_{20/17}$.

(iv) $u \in X_{5/4}$. Since $u \in X_{20/7}$,

$$u \in L^\infty(W^{3/10,20/17}) \cap L^{20/17}(W^{2,20/17}).$$

Observe that, denoting by $D^{3/10}u$ the 3/10-derivatives of u,

$$D^{3/10}u \in L^\infty(L^{20/17}) \cap L^{20/17}(W^{17/10,20/17}) \hookrightarrow L^\infty(L^{20/17}) \cap L^{20/17}(W^{1,60/37}). \tag{53}$$

Applying Lemma 8 to (53) for $(\alpha, r) = (7/10, r)$, $(\beta, p) = (1, 60/37)$, and $(\gamma, q) = (0, 20/17)$, we obtain $(\lambda, r) = (7/10, 150/103)$, and $D^{3/10}u$ satisfies

$$\|D^{3/10}u\|^s_{W^{7/10,150/103}} \le C\|D^{3/10}u\|^{s\,\lambda}_{W^{1,60/37}}\|D^{3/10}u\|^{s(1-\lambda)}_{L^{20/17}}, \tag{54}$$

where expression (54) is integrable in time if $s\,\lambda = 20/17$, and thus $s = 200/119$. Therefore, $u \in L^{200/119}(W^{1,150/103})$ and

$$\nabla u \in L^{200/119}(L^{150/103}).$$

Now, from (44), using (8) (for $p_1 = \infty$, $q_1 = 4$, $p_2 = 20/7$, $q_2 = 60$, and $p = 4$; hence, $q = 12$),

$$\nabla v \in L^\infty(L^4) \cap L^4(L^{12}).$$

Then $\nabla u \cdot \nabla v \in L^{200/119}(L^{300/281}) \cap L^{200/169}(L^{100/77})$, which thanks to (8) applied to $(p_1, q_1) = (200/119, 300/281)$, $(p_2, q_2) = (200/169, 100/77)$ implies $p = q = 5/4$; hence,

$$\nabla u \cdot \nabla v \in L^{5/4}(Q).$$

Since $u\Delta v \in L^{10/7}(Q)$, it is obtained that $\nabla \cdot (u\nabla v) \in L^{5/4}(Q)$. Then, applying Theorem 1 (for $p = 5/4$) to $(1)_1$, it is deduced that $u \in X_{5/4}$.

(v) $u \in X_{4/3}$. Using that $u \in X_{5/4}$,

$$u \in L^\infty(W^{2/5,5/4}) \cap L^{5/4}(W^{2,5/4}).$$

Observe that, denoting by $D^{2/5}u$ the 2/5-derivatives of u,

$$D^{2/5}u \in L^\infty(L^{5/4}) \cap L^{5/4}(W^{8/5,5/4}) \hookrightarrow L^\infty(L^{5/4}) \cap L^{5/4}(W^{1,5/3}). \tag{55}$$

Applying Lemma 8 to (55) for $(\alpha, r) = (3/5, r)$, $(\beta, p) = (1, 5/3)$, and $(\gamma, q) = (0, 5/4)$, we obtain $(\lambda, r) = (3/5, 25/17)$, and $D^{2/5}u$ satisfies

$$\|D^{2/5}u\|^s_{W^{3/5,25/17}} \le C\|D^{2/5}u\|^{s\,\lambda}_{W^{1,5/3}}\|D^{2/5}u\|^{s(1-\lambda)}_{L^{5/4}}, \tag{56}$$

where expression (56) is integrable in time if $s\,\lambda = 5/4$, and thus $s = 25/12$. Therefore, $u \in L^{25/12}(W^{1,25/17})$ and

$$\nabla u \in L^{25/12}(L^{25/17}). \tag{57}$$

From (44), using (8) (for $p_1 = \infty$, $q_1 = 4$, $p_2 = 20/7$, $q_2 = 60$, and $p = 3$, hence $q = 36$), it holds that

$$\nabla v \in L^\infty(L^4) \cap L^3(L^{36});$$

then, from the latter regularity and (57) having

$$\nabla u \cdot \nabla v \in L^{25/12}(L^{100/93}) \cap L^{75/61}(L^{900/637}),$$

which thanks to (8) applied to $(p_1, q_1) = (25/12, 100/93)$, $(p_2, q_2) = (75/61, 900/637)$ implies $p = q = 4/3$; hence,

$$\nabla u \cdot \nabla v \in L^{4/3}(Q).$$

Since $u\Delta v \in L^{10/7}(Q)$, it is obtained that $\nabla \cdot (u\nabla v) \in L^{4/3}(Q)$. Then, applying Theorem 1 (for $p = 4/3$) to $(1)_1$, it is deduced that $u \in X_{4/3}$.

(vi) $u \in X_{10/7}$. Since $u \in X_{4/3}$,

$$u \in L^\infty(W^{1/2,4/3}) \cap L^{4/3}(W^{2,4/3}).$$

Observe that, denoting by $D^{1/2}u$ the 1/2-derivatives of u,

$$D^{1/2}u \in L^\infty(L^{4/3}) \cap L^{4/3}(W^{3/2,4/3}) \hookrightarrow L^\infty(L^{4/3}) \cap L^{4/3}(W^{1,12/7}). \tag{58}$$

Applying Lemma 8 to (58) for $(\alpha, r) = (1/2, r)$, $(\beta, p) = (1, 12/7)$, and $(\gamma, q) = (0, 4/3)$, we obtain $(\lambda, r) = (1/2, 3/2)$, and $D^{1/2}u$ satisfies

$$\|D^{1/2}u\|^s_{W^{1/2,3/2}} \le C\|D^{1/2}u\|^{s\,\lambda}_{W^{1,12/7}}\|D^{1/2}u\|^{s(1-\lambda)}_{L^{4/3}}, \tag{59}$$

where expression (59) is integrable in time if $s\,\lambda = 4/3$, and thus $s = 8/3$. Therefore, $u \in L^{8/3}(W^{1,3/2})$ and

$$\nabla u \in L^{8/3}(L^{3/2}).$$

Again using that $\nabla v \in L^3(L^{36})$, it is obtained that

$$\nabla v \in L^\infty(L^4) \cap L^3(L^{36})$$

and $\nabla u \cdot \nabla v \in L^{8/3}(L^{12/11}) \cap L^{24/17}(L^{36/25})$, which thanks to (8) applied to $(p_1, q_1) = (8/3, 12/11)$, $(p_2, q_2) = (24/17, 36/25)$ implies $p = q = 10/7$; hence,

$$\nabla u \cdot \nabla v \in L^{10/7}(Q).$$

Since $u\Delta v \in L^{10/7}(Q)$, it is obtained that $\nabla \cdot (u\nabla v) \in L^{10/7}(Q)$. Then, applying Theorem 1 (for $p = 10/7$) to $(1)_1$, it is deduced that $u \in X_{10/7}$.

(vii) $u \in X_{20/13}$. Since $u \in X_{10/7}$,

$$u \in L^\infty(W^{3/5,10/7}) \cap L^{10/7}(W^{2,10/7}) \hookrightarrow L^\infty(L^2) \cap L^{10/7}(L^{30}). \tag{60}$$

In particular, it follows that

$$\nabla u \in L^{10/7}(W^{1,10/7}) \hookrightarrow L^{10/7}(L^{30/11}). \tag{61}$$

Observe that, denoting by $D^{3/5}u$ the 3/5-derivatives of u,

$$D^{3/5}u \in L^\infty(L^{10/7}) \cap L^{10/7}(W^{7/5,10/7}) \hookrightarrow L^\infty(L^{10/7}) \cap L^{10/7}(W^{1,30/17}). \tag{62}$$

Applying Lemma 8 to (62) for $(\alpha, r) = (2/5, r)$, $(\beta, p) = (1, 30/17)$, and $(\gamma, q) = (0, 10/7)$, we obtain $(\lambda, r) = (2/5, 150/97)$, and $D^{3/5}u$ satisfies

$$\|D^{3/5}u\|^s_{W^{2/5,150/97}} \le C\|D^{3/5}u\|^{s\,\lambda}_{W^{1,30/17}}\|D^{3/5}u\|^{s(1-\lambda)}_{L^{10/7}}, \tag{63}$$

where expression (63) is integrable in time if $s\,\lambda = 10/7$, and thus $s = 25/7$. From (63), it can be deduced that

$$D^{3/5}u \in L^{25/7}(W^{2/5,150/97}),$$

which implies

$$\nabla u = D^{2/5}(D^{3/5}u) \in L^{25/7}(L^{150/97}). \tag{64}$$

Therefore, thanks to (60), (61), and (64), $u \in L^{25/7}(W^{1,150/97}) \cap L^{10/7}(L^{30/11})$. Applying Lemma 5 to (60) and using the previous regularity, it is deduced that

$$\begin{cases} u \in L^\infty(W^{3/5,10/7}) \cap L^{10/7}(W^{2,10/7}) \hookrightarrow L^\infty(L^2) \cap L^{10/7}(L^{30}) \hookrightarrow L^{10/3}(Q), \\ \nabla u \in L^{25/7}(L^{150/97}) \cap L^{10/7}(L^{30/11}). \end{cases} \tag{65}$$

This time, (8) is used in $(65)_2$ (for $(p_1, q_1) = (25/7, 150/97)$ and $(p_2, q_2) = (10/7, 30/11)$ and $p = q = 2$), obtaining

$$\nabla u \in L^2(Q).$$

The latter regularity, (65), (45), and the fact that $\Delta v \in L^{20/7}(Q)$ imply

$$u\Delta v,\ \nabla u \cdot \nabla v \in L^{20/13}(Q).$$

Then, applying Theorem 1 (for $p = 20/13$) to $(1)_1$, it is deduced that $u \in X_{20/13}$.

(viii) $u \in L^\infty(L^2) \cap L^2(H^1)$. From (9), it is known that $W^{7/10,20/13}(\Omega) \hookrightarrow H^{1/4}(\Omega)$ and $W^{2,20/13}(\Omega) \hookrightarrow H^{31/20}(\Omega)$. Therefore, from $u \in X_{20/13}$, the regularity

$$u \in L^\infty(H^{1/4}) \cap L^{20/13}(H^{31/20})$$

can be deduced. Moreover, from Lemma 6 for $(p_1, s_1) = (\infty, 1/4)$, $(p_2, s_2) = (20/13, 31/20)$, one has that $u \in L^2(H^{5/4}) \hookrightarrow L^2(H^1)$. Therefore, from the latter regularity and $(65)_1$, it can be deduced that

$$u \in L^\infty(L^2) \cap L^2(H^1) \hookrightarrow L^{10/3}(Q). \tag{66}$$

Step 3: $(u, v) \in X_{5/3} \times X_{10/3}$. In particular, $u \in L^5(Q)$ and $\nabla u \in L^{20/9}(Q)$.

From (43), (66), and the fact that $f \in L^4(Q_c)$, it implies that $u + f\,v\,1_{\Omega_c} \in L^{10/3}(Q)$. Then, applying Theorem 1 (for $p = 10/3$) to $(1)_2$, one has $v \in X_{10/3}$. In particular, from Lemma 7 (for $p_1 = p_2 = 10/3$, $q_1 = 6$, $r = 1$, and $p = q = 10$), $\nabla v \in L^\infty(L^6) \cap L^{10/3}(W^{1,10/3}) \hookrightarrow L^{10}(Q)$ is obtained. Then, using that $(u, \Delta v) \in L^{10/3}(Q) \times L^{10/3}(Q)$, $\nabla v \in L^{10}(Q)$ and taking into account that $\nabla u \in L^2(Q)$, it holds that

$$\nabla \cdot (u\nabla v) = u\Delta v + \nabla u \cdot \nabla v \in L^{5/3}(Q).$$

Thus, applying Theorem 1 (for $p = 5/3$) to $(1)_1$, $u \in X_{5/3}$ is obtained. Moreover, from Sobolev embeddings and again Lemma 7 (for $p_1 = p_2 = 5/3$, $q_1 = 3$, $r = 2$, and $p = q = 5$), the regularity

$$u \in L^\infty(L^3) \cap L^{5/3}(W^{2,5/3}) \hookrightarrow L^5(Q) \tag{67}$$

holds.

From (9), the embeddings $W^{4/5,5/3}(\Omega) \hookrightarrow H^{1/2}(\Omega)$ and $W^{2,5/3}(\Omega) \hookrightarrow H^{17/10}(\Omega)$ hold. Thus, since $u \in X_{5/3}$, one has

$$u \in L^\infty(H^{1/2}) \cap L^{5/3}(H^{17/10}).$$

Moreover, from Lemma 6 (for $(p_1, s_1) = (\infty, 1/2)$ and $(p_2, s_2) = (5/3, 17/10)$), it can be deduced that $u \in L^{20/9}(H^{7/5})$, and in particular $\nabla u \in L^{20/9}(H^{2/5}) \hookrightarrow L^{20/9}(Q)$.

Step 4: $(u, v) \in X_2 \times X_4$.

From (43), (67), and using that $f \in L^4(Q_c)$, one has $u + f\,v\,1_{\Omega_c} \in L^4(Q)$. Then, applying Theorem 1 (for $p = 4$) to $(1)_2$, it can be deduced that $v \in X_4$ and satisfies the estimate

(68)
$$\begin{aligned}\|v\|_{X_4} &\le C(\|u + fv\|_{L^4(Q)} + \|v_0\|_{W_{\mathbf{n}}^{3/2,4}})\\ &\le C(\|u\|_{L^4(Q)} + \|f\|_{L^4(Q)}\|v\|_{L^\infty(Q)} + \|v_0\|_{W_{\mathbf{n}}^{3/2,4}})\\ &\le C_0(\|u_0\|_{W^{4/5,5/3}}, \|v_0\|_{W_{\mathbf{n}}^{3/2,4}}, \|f\|_{L^4(Q)}).\end{aligned}$$

In particular, by Sobolev embeddings and Lemma 7 (for $p_1 = p_2 = 4$, $q_1 = 12$, $r = 1$, hence $p = q = 20$), the regularity $\nabla v \in L^\infty(L^{12}) \cap L^4(W^{1,4}) \hookrightarrow L^{20}(Q)$ is deduced.

Now, using that $(u, \Delta v) \in L^5(Q) \times L^4(Q)$ and $(\nabla u, \nabla v) \in L^{20/9}(Q) \times L^{20}(Q)$, one has

$$\nabla \cdot (u\nabla v) = u\Delta v + \nabla u \cdot \nabla v \in L^2(Q).$$

Therefore, applying Theorem 1 (for $p = 2$) to $(1)_1$, it is deduced that $u \in X_2$ and

$$\begin{aligned} \|u\|_{X_2} &\le C(\|u\|_{L^5(Q)}\|\Delta v\|_{L^4(Q)} + \|\nabla u\|_{L^{20/9}(Q)}\|\nabla v\|_{L^{20}(Q)} + \|u_0\|_{H^1}) \\ &\le C_1(\|u_0\|_{H^1}, \|v_0\|_{W^{3/2,4}_{\mathbf{n}}}, \|f\|_{L^4(Q)}). \end{aligned} \tag{69}$$

Step 5: $u \in X_4$.

Observe that the regularity $v \in X_4$ cannot be improved because $f \in L^4(Q_c)$; hence, $f\, v\, 1_{\Omega_c} \in L^4(Q)$ and no more. However, the regularity for u can increase from X_2 to X_4.

(i) $u \in X_{20/7}$. Since $(u, v) \in X_2 \times X_4$, it is deduced that $u \in L^{10}(Q)$, $\nabla u \in L^{10/3}(Q)$, $\Delta v \in L^4(Q)$, and, by using Lemma 7, one has

$$\nabla v \in L^\infty(W^{1/2,4}) \cap L^4(W^{1,4}) \hookrightarrow L^\infty(L^{12}) \cap L^4(W^{1,4}) \hookrightarrow L^{20}(Q). \tag{70}$$

Therefore, $\nabla \cdot (u\nabla v) = u\,\Delta v + \nabla u \cdot \nabla v \in L^{20/7}(Q)$. Again from Theorem 1, it can be deduced that $u \in X_{20/7}$ and

$$\begin{aligned} \|u\|_{X_{20/7}} &\le C(\|u\|_{L^{10}(Q)}\|\Delta v\|_{L^4(Q)} + \|\nabla u\|_{L^{10/3}(Q)}\|\nabla v\|_{L^{20}(Q)} + \|u_0\|_{W^{13/10,20/7}}) \\ &\le C_1(\|u_0\|_{W^{13/10,20/7}}, \|v_0\|_{W^{3/2,4}_{\mathbf{n}}}, \|f\|_{L^4(Q)}). \end{aligned} \tag{71}$$

(ii) $u \in X_4$. Since $v \in X_4$, $\nabla v \in L^{20}(Q)$ and $\Delta v \in L^4(Q)$. From $u \in X_{20/7}$, the Sobolev embedding $u \in L^\infty(W^{13/10,20/7}) \hookrightarrow L^\infty(Q)$ can be deduced. Moreover, using Lemma 7, it is known that $\nabla u \in L^\infty(W^{3/10,20/7}) \cap L^{20/7}(W^{1,20/7}) \hookrightarrow L^{20/3}(Q)$. Therefore,

$$\nabla \cdot (u\nabla v) = u\,\Delta v + \nabla u \cdot \nabla v \in L^4(Q).$$

Thus, Theorem 1 implies that $u \in X_4$ and

$$\begin{aligned} \|u\|_{X_4} &\le C(\|u\|_{L^\infty(Q)}\|\Delta v\|_{L^4(Q)} + \|\nabla u\|_{L^{20/3}(Q)}\|\nabla v\|_{L^{20}(Q)} + \|u_0\|_{W^{3/2,4}}) \\ &\le C_1(\|u_0\|_{W^{3/2,4}_{\mathbf{n}}}, \|v_0\|_{W^{3/2,4}_{\mathbf{n}}}, \|f\|_{L^4(Q)}). \end{aligned} \tag{72}$$

Finally, observe that estimate (40) follows from (68) and (72).

5. The optimal control problem. In this section, the statement of the bilinear control problem is established. Following [7, 8], the control problem in such a way that any admissible state is a strong solution of (1)–(3) is formulated. It is supposed that

$$\mathcal{F} \subset L^4(Q_c) := L^4(0, T; L^4(\Omega_c)) \quad \text{is a nonempty, closed, and convex set,} \tag{73}$$

where $\Omega_c \subset \Omega$ is the control domain. Note that the physically relevant case where pointwise control constraints are imposed is a particular case in this analysis because the set

$$\mathcal{F} = \{f \in L^4(0,T;L^4(\Omega_c)) : -\infty < a \le f(t,x) \le b < +\infty \ a.e.\ (t,x) \in (0,T) \times \Omega_c\}$$

is a nonempty closed convex set in $L^4(0,T;L^4(\Omega_c))$.

Consider $(u_0, v_0) \in W_{\mathbf{n}}^{3/2,4}(\Omega)^2$ the initial data with $u_0 \ge 0$ and $v_0 \ge 0$ in Ω and the function $f \in \mathcal{F}$ describing the bilinear control acting on the v-equation.

Now the following constrained minimization problem related to system (1)–(3) is defined:

(74)

$$\begin{cases} \text{Find } (u,v,f) \in X_4 \times X_4 \times \mathcal{F} \text{ such that the functional} \\ J(u,v,f) := \dfrac{\gamma_u}{20/7} \displaystyle\int_0^T \|u(t) - u_d(t)\|_{L^{20/7}(\Omega)}^{20/7} dt + \dfrac{\gamma_v}{2} \displaystyle\int_0^T \|v(t) - v_d(t)\|_{L^2(\Omega)}^2 dt \\ \qquad\qquad + \dfrac{\gamma_f}{4} \displaystyle\int_0^T \|f(t)\|_{L^4(\Omega_c)}^4 dt \\ \text{is minimized, subject to } (u,v,f), \text{ and satisfies the PDE system (1)–(3).} \end{cases}$$

Here $(u_d,\ v_d) \in L^{20/7}(Q) \times L^2(Q)$ represents the desired states, and the real numbers γ_u, γ_v, and γ_f measure the cost of the states and control, respectively. These numbers satisfy

$$\gamma_u > 0 \quad \text{and} \quad \gamma_v, \gamma_f \ge 0. \tag{75}$$

In fact, it is assumed that either $\gamma_f > 0$ or $\mathcal{F}$ is bounded in $L^4(Q_c)$.

The admissible set for the optimal control problem (74) is defined by

$$\mathcal{S}_{ad} = \{s = (u,v,f) \in X_4 \times X_4 \times \mathcal{F} : s \text{ is a strong solution of (1)–(3) in } (0,T)\}.$$

The functional J defined in (74) describes the deviation of the cell density u and the chemical concentration v from a desired cell density u_d and chemical concentration v_d, respectively, plus the cost of the control measured in the L^4-norm. Since there is no existence result of global-in-time strong solutions of (1)–(3), it is necessary to choose a suitable objective functional, considering particularly the $L^{20/7}(Q)$-norm for u. Consequently, if (u,v) is a weak solution of (1)–(3) in $(0,T)$ such that $J(u,v,f) < +\infty$, then $u \in L^{20/7}(Q)$ and, by Theorem 7, $(u,v) \in X_4 \times X_4$ is a strong solution of (1)–(3) in $(0,T)$. In what follows, the hypothesis will be assumed:

$$\mathcal{S}_{ad} \neq \emptyset. \tag{76}$$

Remark 5. The reason for choosing the first term of the objective functional in the $L^{20/7}$-norm is that any weak solution of (1)–(3) such that $J(u,v,f) < +\infty$ satisfies that $u \in L^{20/7}(Q)$, and therefore, by virtue of Theorem 7, (u,v) is the unique solution of (1)–(3) in the sense of Definition 3. Thus, the admissible states of problem (74) to the strong solutions of (1)–(3) are reduced. With this formulation, the existence of a global optimal solution will be proved, and the optimality conditions associated to any local optimal solution will be derived.

Remark 6. The hypothesis (76) holds when the control acts on the whole domain Ω, that is, $\Omega_c = \Omega$, and the initial chemical concentration is strictly positive, $v_0 \ge \alpha > 0$ in Ω. In this case, we will furnish a particular $(u,v,f) \in \mathcal{S}_{ad}$.

Indeed, by applying Theorem 1 for $p = 4$, one first considers $v \in X_4$ as the global-in-time strong solution of the heat problem

$$\partial_t v - \Delta v = 0 \text{ in } Q, \quad v(0,\cdot) = v_0 \text{ in } \Omega, \quad \frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } (0,T)\times\partial\Omega.$$

This v is strictly positive because $v \geq v_0 \geq \alpha > 0$ in Q. Second, given this $v \in X_4$, we define $u \in X_4$ as the strong solution of the linear parabolic problem

$$\partial_t u - \Delta u - \nabla\cdot(u\nabla v) = 0, \quad u(0,\cdot) = u_0 \text{ in } \Omega, \quad \frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } (0,T)\times\partial\Omega.$$

In fact, the existence (and uniqueness) of $u \in X_{20/11}$ using Theorem 2 for $\mathcal{X} = X$ (X defined in (4)) and $\mathcal{Y} = X_{20/11}$ can be proved. The embedding from $X_{20/11}$ into $L^{20/3}(Q)$ (see (109) below) implies in particular that the regularity criterion $u \in L^{20/7}(Q)$ (given in Theorem 7) holds. Therefore, it can be deduced that $u \in X_4$.

Finally, if f is defined such that $v = u + f\,v$ (i.e., $f = 1 - u/v$), which is well defined and regular because $v \geq v_0 \geq \alpha > 0$ in Q, then $(u, v, f) \in \mathcal{S}_{ad}$.

5.1. Existence of global optimal solution.

Definition 4. *An element $(\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ will be called a global optimal solution of problem* (74) *if*

$$J(\tilde{u}, \tilde{v}, \tilde{f}) = \min_{(u,v,f)\in\mathcal{S}_{ad}} J(u, v, f). \tag{77}$$

Theorem 8. *Let u_0, $v_0 \in W^{3/2,4}_{\mathbf{n}}(\Omega)$ with $u_0 \geq 0$ and $v_0 \geq 0$ in Ω. Assuming that either $\gamma_f > 0$ or $\mathcal{F}$ is bounded in $L^4(Q_c)$ and hypothesis* (76)*, the bilinear optimal control problem* (74) *has at least one global optimal solution $(\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$.*

Proof. From hypothesis (76), $\mathcal{S}_{ad} \neq \emptyset$. Let $\{s_m\}_{m\in\mathbb{N}} := \{(u_m, v_m, f_m)\}_{m\in\mathbb{N}} \subset \mathcal{S}_{ad}$ be a minimizing sequence of J, that is, $\lim_{m\to+\infty} J(s_m) = \inf_{s\in\mathcal{S}_{ad}} J(s)$. Then, by definition of $\mathcal{S}_{ad}$, for each $m \in \mathbb{N}$, s_m satisfies system (1) a.e. $(t, x) \in Q$.

From the definition of J and the assumption $\gamma_f > 0$ or $\mathcal{F}$ is bounded in $L^4(Q_c)$, it follows that

$$\{f_m\}_{m\in\mathbb{N}} \text{ is bounded in } L^4(Q_c) \tag{78}$$

and

$$\{u_m\}_{m\in\mathbb{N}} \text{ is bounded in } L^{20/7}(Q).$$

From (40) there exists a positive constant C, independent of m, such that

$$\|(u_m, v_m)\|_{X_4\times X_4} \leq C. \tag{79}$$

Therefore, from (78), (79), and taking into account that $\mathcal{F}$ is a closed convex subset of $L^4(Q_c)$ (hence is weakly closed in $L^4(Q_c)$), it is deduced that there exists $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in X_4 \times X_4 \times \mathcal{F}$ such that, for some subsequence of $\{s_m\}_{m\in\mathbb{N}}$, still denoted by $\{s_m\}_{m\in\mathbb{N}}$, the following convergences hold as $m \to +\infty$:

$$u_m \to \tilde{u} \quad \text{weakly in } L^4(W^{2,4}) \text{ and weakly* in } L^\infty(W^{3/2,4}_{\mathbf{n}}); \tag{80}$$

$$v_m \to \tilde{v} \quad \text{weakly in } L^4(W^{2,4}) \text{ and weakly* in } L^\infty(W^{3/2,4}_{\mathbf{n}}); \tag{81}$$

$$\partial_t u_m \to \partial_t \tilde{u} \quad \text{weakly in } L^4(Q); \tag{82}$$

$$\partial_t v_m \to \partial_t \tilde{v} \quad \text{weakly in } L^4(Q); \tag{83}$$

$$f_m \to \tilde{f} \quad \text{weakly in } L^4(Q_c) \text{ and } \tilde{f} \in \mathcal{F}. \tag{84}$$

From (80)–(83), Theorems 3 and 4, and using Sobolev embedding, one has

$$(u_m, v_m) \to (\tilde u, \tilde v) \quad \text{strongly in } (C([0,T]; L^q) \cap L^4(W^{1,q}))^2 \ \ \forall q < +\infty. \tag{85}$$

In particular, the limit of the nonlinear terms of (1) can be controlled as follows:

$$\nabla \cdot (u_m \nabla v_m) \to \nabla \cdot (\tilde u \nabla \tilde v) \quad \text{weakly in } L^{10/3}(Q), \tag{86}$$

$$f_m v_m 1_{\Omega_c} \to \tilde f\, \tilde v\, 1_{\Omega_c} \quad \text{weakly in } L^4(Q). \tag{87}$$

Moreover, from (85) it implies that $(u_m(0), v_m(0))$ converges to $(\tilde u(0), \tilde v(0))$ in $L^q(\Omega) \times L^q(\Omega)$, and since $u_m(0) = u_0$, $v_m(0) = v_0$, it is deduced that $\tilde u(0) = u_0$ and $\tilde v(0) = v_0$. Thus, $\tilde s$ satisfies the initial conditions given in (2). Therefore, considering the convergences (80)–(87) and taking the limit in (1) replacing (u, v, f) by (u_m, v_m, f_m) as m goes to $+\infty$, it is possible to conclude that $\tilde s = (\tilde u, \tilde v, \tilde f)$ is a solution of the system (1) pointwisely, that is, $\tilde s \in \mathcal{S}_{ad}$. Therefore,

$$\lim_{m\to+\infty} J(s_m) = \inf_{s\in\mathcal{S}_{ad}} J(s) \le J(\tilde s). \tag{88}$$

Additionally, since J is lower semicontinuous on $\mathcal{S}_{ad}$, one has

$$J(\tilde s) \le \liminf_{m\to+\infty} J(s_m),$$

which, jointly with (88), implies (77). □

5.2. Optimality system related to local optimal solutions. The first-order necessary optimality conditions for a local optimal solution $(\tilde u, \tilde v, \tilde f)$ of problem (74) will be derived, applying a Lagrange multiplier theorem. The argument is based on a generic result given by Zowe and Kurcyusz [38] (see also [35, Chapter 6] for more details) on the existence of Lagrange multiplier in Banach spaces. In order to introduce the concepts and results given in [38], the following optimization problem will be considered:

$$\min_{s\in\mathbb{M}} J(s) \ \text{ subject to } G(s) = 0. \tag{89}$$

Here, $J : \mathbb{X} \to \mathbb{R}$ is a functional, $G : \mathbb{X} \to \mathbb{Y}$ is an operator, $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces, and $\mathbb{M}$ is a nonempty closed and convex subset of $\mathbb{X}$. The admissible set for problem (89) is defined by

$$\mathcal{S} = \{s \in \mathbb{M} : G(s) = 0\}.$$

Definition 5 (Lagrangian). *The functional* $\mathcal{L} : \mathbb{X} \times \mathbb{Y}' \to \mathbb{R}$, *given by*

$$\mathcal{L}(s, \xi) = J(s) - \langle \xi, G(s)\rangle_{\mathbb{Y}'}, \tag{90}$$

is called the Lagrangian functional related to problem (89).

Definition 6 (Lagrange multiplier). *Let* $\tilde s \in \mathcal{S}$ *be a local optimal solution for problem* (89). *Suppose that* J *and* G *are Fréchet differentiable in* $\tilde x$, *with derivatives* $J'(\tilde s)$ *and* $G'(\tilde s)$, *respectively. Then any* $\xi \in \mathbb{Y}'$ *is called a Lagrange multiplier for* (89) *at the point* $\tilde s$ *if*

$$\begin{cases} \langle \xi, G(\tilde s)\rangle_{\mathbb{Y}'} = 0, \\ \mathcal{L}'(\tilde s, \xi)[r] = J'(\tilde s)[r] - \langle \xi, G'(\tilde s)[r]\rangle_{\mathbb{Y}'} \ge 0 \quad \forall r \in \mathcal{C}(\tilde s), \end{cases} \tag{91}$$

where $\mathcal{C}(\tilde s) = \{\theta(s - \tilde s) : s \in \mathbb{M}, \theta \ge 0\}$ *is the conical hull of* $\tilde s$ *in* $\mathbb{M}$.

DEFINITION 7. *Let $\tilde{s} \in \mathcal{S}$ be a local optimal solution for problem* (89). *It will be said that $\tilde{s}$ is a regular point if*

$$G'(\tilde{s})[\mathcal{C}(\tilde{s})] = \mathbb{Y}.$$

THEOREM 9 ([35, Theorem 6.3, p. 330], [38, Theorem 3.1]). *Let $\tilde{s} \in \mathcal{S}$ be a local optimal solution for problem* (89). *Suppose that J is a Fréchet differentiable function and G is continuous Fréchet differentiable. If $\tilde{s}$ is a regular point, then the set of Lagrange multipliers for* (89) *at $\tilde{s}$ is nonempty.*

Now the optimal control problem (74) will be reformulated in the abstract setting (89). The Banach spaces

$$\mathbb{X} := \widehat{X}_4 \times \widehat{X}_4 \times L^4(Q_c), \ \mathbb{Y} := L^4(Q) \times L^4(Q),$$

are considered, where

$$\widehat{X}_4 = \{u \in X_4 \, : \, u(0) = 0\}$$

and the operator $G = (G_1, G_2) : \mathbb{X} \to \mathbb{Y}$, where

$$G_1 : \mathbb{X} \to L^4(Q), \ G_2 : \mathbb{X} \to L^4(Q),$$

are defined at each point $s = (u, v, f) \in \mathbb{X}$ by

$$\begin{cases} G_1(s) = \partial_t u - \Delta u - \nabla \cdot (u \nabla v), \\ G_2(s) = \partial_t v - \Delta v + v - u - f\, v\, 1_{\Omega_c}. \end{cases}$$

Thus, the optimal control problem (74) is reformulated as

$$\min_{s \in \mathbb{M}} J(s) \quad \text{subject to} \quad G(s) = \mathbf{0}, \tag{92}$$

where

$$\mathbb{M} := (\hat{u}, \hat{v}, \hat{f}) + \widehat{X}_4 \times \widehat{X}_4 \times (\mathcal{F} - \hat{f}),$$

where $(\hat{u}, \hat{v}, \hat{f})$ is a global strong solution of (1)–(3) and $\mathcal{F}$ is defined in (73).

Remark 7. From Definition 5, it is deduced that the Lagrangian associated to optimal control problem (92) is the functional $\mathcal{L} : \mathbb{X} \times L^{4/3}(Q) \times L^{4/3}(Q) \to \mathbb{R}$ given by

$$\mathcal{L}(s, \lambda, \eta) = J(s) - \langle \lambda, G_1(s) \rangle_{L^{4/3}} - \langle \eta, G_2(s) \rangle_{L^{4/3}}.$$

It can be observed that $\mathbb{M}$ is a closed convex subset of $\mathbb{X}$ and that the set of admissible solutions of control problem (92) is

$$\mathcal{S}_{ad} = \{s = (u, v, f) \in \mathbb{M} \, : \, G(s) = \mathbf{0}\}. \tag{93}$$

Concerning the differentiability of the constraint operator G and the functional J, one has the following results.

LEMMA 10. *The functional $J : \mathbb{X} \to \mathbb{R}$ is Fréchet differentiable, and the Fréchet derivative of J in $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathbb{X}$ in the direction $r = (U, V, F) \in \mathbb{X}$ is*

(94)

$$J'(\tilde{s})[r] = \gamma_u \int_0^T \!\! \int_\Omega \operatorname{sgn}(\tilde{u} - u_d) |\tilde{u} - u_d|^{13/7} U + \gamma_v \int_0^T \!\! \int_\Omega (\tilde{v} - v_d) V + \gamma_f \int_0^T \!\! \int_{\Omega_c} (\tilde{f})^3 F.$$

LEMMA 11. *The operator $G : \mathbb{X} \to \mathbb{Y}$ is continuous Fréchet differentiable, and the Fréchet derivative of G in $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathbb{X}$ in the direction $r = (U, V, F) \in \mathbb{X}$ is the linear operator $G'(\tilde{s})[r] = (G'_1(\tilde{s})[r], G'_2(\tilde{s})[r])$ defined by*

$$\text{(95)} \qquad \begin{cases} G'_1(\tilde{s})[r] = \partial_t U - \Delta U - \nabla\cdot(U\nabla\tilde{v}) - \nabla\cdot(\tilde{u}\nabla V), \\ G'_2(\tilde{s})[r] = \partial_t V - \Delta V + V - U - \tilde{f}\, V\, 1_{\Omega_c} - F\tilde{v}. \end{cases}$$

The aim is to prove the existence of Lagrange multipliers, which is guaranteed if a local optimal solution of problem (92) is a regular point of operator G (by virtue of Theorem 9).

Remark 8. From Definition 7 it is concluded that $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ is a regular point if for any $(g_u, g_v) \in \mathbb{Y}$ there exists $r = (U, V, F) \in \widehat{X}_4 \times \widehat{X}_4 \times \mathcal{C}(\tilde{f})$ such that

$$G'(\tilde{s})[r] = (g_u, g_v),$$

where $\mathcal{C}(\tilde{f}) := \{\theta(f - \tilde{f}) \,:\, \theta \geq 0,\, f \in \mathcal{F}\}$ is the conical hull of $\tilde{f}$ in $\mathcal{F}$.

LEMMA 12. *Let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ ($\mathcal{S}_{ad}$ defined in* (93)*). Then $\tilde{s}$ is a regular point.*

Proof. For a fixed point $(\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$, let $(g_u, g_v) \in \mathbb{Y} = L^4(Q)^2$. Since $0 \in \mathcal{C}(\tilde{f}) = \{\theta(f - \tilde{f}) \,:\, \theta \geq 0,\, f \in \mathcal{F}\}$, it suffices to show the existence of $(U, V) \in X_4 \times X_4$ solving the linear problem

$$\text{(96)} \qquad \begin{cases} \partial_t U - \Delta U - \nabla\cdot(U\nabla\tilde{v}) - \nabla\cdot(\tilde{u}\nabla V) = g_u \quad \text{in } Q, \\ \partial_t V - \Delta V + V - U - \tilde{f}\, V\, 1_{\Omega_c} = g_v \quad \text{in } Q, \\ U(0) = 0,\ V(0) = 0 \quad \text{in } \Omega, \\ \dfrac{\partial U}{\partial \mathbf{n}} = 0,\ \dfrac{\partial V}{\partial \mathbf{n}} = 0 \quad \text{on } (0,T)\times\partial\Omega. \end{cases}$$

In order to prove the existence of solution of (96), Theorem 2 (Leray–Schauder fixed-point theorem) will be utilized for $\mathcal{X} = X \times X$ (X is defined in (4)) and $\mathcal{Y} = X_{20/11} \times X_{20/11}$. Therefore, the operator

$$\text{(97)} \qquad S : (\overline{U}, \overline{V}) \in X \times X \mapsto (U, V) \in X_{20/11} \times X_{20/11}$$

is considered, where (U, V) is the solution of the decoupled problem

$$\text{(98)} \qquad \begin{cases} \partial_t U - \Delta U - \nabla\cdot(\tilde{u}\nabla V) = \nabla\cdot(\overline{U}\nabla\tilde{v}) + g_u \quad \text{in } Q, \\ \partial_t V - \Delta V + V = \overline{U} + \tilde{f}\overline{V} 1_{\Omega_c} + g_v \quad \text{in } Q, \end{cases}$$

endowed with the corresponding initial and boundary conditions.

Step 1: In order to prove Lemma 1 for the operator S defined in (97), starting from $(\overline{U}, \overline{V})$, one first finds V and later finds U. Indeed, from Corollary 1, it is known that $\overline{U}, \overline{V} \in L^{10/3}(Q)$; hence, $f\,\bar{V} 1_{\Omega_c} \in L^{20/11}(Q)$. Applying Theorem 1 (for $p = 20/11$) to $(98)_2$, it is deduced that $V \in X_{20/11}$ and

$$\text{(99)} \qquad \begin{aligned} \|V\|_{X_{20/11}} &\leq C\left(\|\overline{U} + \tilde{f}\overline{V} 1_{\Omega_c} + g_v\|_{L^{20/11}(Q)}\right) \\ &\leq C\left(\|\overline{U}\|_X + \|\tilde{f}\|_{L^4(Q)}\|\overline{V}\|_X + \|g_v\|_{L^4(Q)}\right). \end{aligned}$$

Owing to Remark 4, it is known that $\tilde{v} \in X_4$ implies $\nabla\tilde{v} \in L^{20}(Q)$, and it can be deduced that

$$\text{(100)} \qquad \nabla\cdot(\overline{U}\nabla\tilde{v}) = \overline{U}\,\Delta\tilde{v} + \nabla\overline{U}\cdot\nabla\tilde{v} \in L^{20/11}(Q).$$

Observe that, using (9) and Lemma 6,

(101)
$$V \in X_{20/11} \hookrightarrow L^\infty(W^{9/10,20/11}) \cap L^{20/11}(W^{2,20/11}) \hookrightarrow L^\infty(H^{3/4}) \cap L^{20/11}(H^{37/20}).$$

In particular, from (101), $V \in X_{20/11} \hookrightarrow L^{20/7}(H^{29/20})$. Therefore, using again Lemma 6,

$$\nabla V \in L^{20/7}(H^{9/20}) \hookrightarrow L^{20/7}(Q), \tag{102}$$

and thus

$$\nabla \cdot (\tilde{u} \nabla V) = \tilde{u}\, \Delta V + \nabla \tilde{u} \cdot \nabla V \in L^{20/11}(Q) + L^{5/2}(Q) \tag{103}$$

thanks to (41) and (42).

Therefore, from (100), (103), and Theorem 1, it is deduced that $U \in X_{20/11}$ and

$$\begin{aligned} \|U\|_{X_{20/11}} &\leq C\, \|\nabla \cdot (\overline{U} \nabla \tilde{v}) + \nabla \cdot (\tilde{u} \nabla V) + g_u\|_{L^{20/11}(Q)} \\ &\leq C\, \left(\|\overline{U}\|_X \|\tilde{v}\|_{X_4} + \|V\|_{X_{20/11}} \|\tilde{u}\|_{X_4} \right). \end{aligned} \tag{104}$$

Finally, from (99) and (104), it is deduced that bounded sets in $X \times X$ are mapped in bounded sets in $X_{20/11} \times X_{20/11}$.

Step 2: From Corollary 2 it can be deduced that $X_{20/11}$ is compactly embedded in X. Therefore, Lemma 2 is proved in this case.

Step 3: In particular, using the argument of Lemma 14 (see Appendix A), it is not difficult to prove the continuity of S from $X \times X$ to itself.

Step 4: Now the aim is to show that the set $S_\alpha := \{(U,V) \in X_{20/11} \times X_{20/11} : (U,V) = \alpha S(U,V) \text{ for some } \alpha \in [0,1]\}$ is bounded in $X \times X$ (with respect to α). Indeed, if $(U,V) \in S_\alpha$, then $(U,V) \in X_{20/11} \times X_{20/11}$ and solves the coupled linear problem

$$\left\{ \begin{array}{r} \partial_t U - \Delta U - \nabla \cdot (\tilde{u} \nabla V) = \alpha \nabla \cdot (U \nabla \tilde{v}) + \alpha g_u \ \text{ in } Q, \\ \partial_t V - \Delta V + V = \alpha U + \alpha \tilde{f} V 1_{\Omega_c} + \alpha g_v \ \text{ in } Q, \end{array} \right. \tag{105}$$

with the corresponding initial and boundary conditions. Then, testing $(105)_1$ by U, one obtains

(106)
$$\frac{d}{dt}\|U\|^2 + \|\nabla U\|^2 \leq C\, \alpha^4 \left(1 + \|\nabla \tilde{v}\|_{L^6}^4\right) \|U\|^2 + C\, \|\tilde{u}\|_{L^\infty}^2 \|\nabla V\|^2 + \alpha \left(\|g_u\|^2 + \|U\|^2\right).$$

Now, testing $(105)_2$ by V, it holds that

$$\frac{d}{dt}\|V\|^2 + \|V\|_{H^1}^2 \leq C\, \alpha^4 \|f\|_{L^2}^4 \|V\|^2 + \alpha \left(\|g_v\|^2 + \|U\|^2 + \|V\|^2\right). \tag{107}$$

Considering an adequate constant $K > 2C\, \|\tilde{u}\|_{L^\infty}^2$ (recall that $\tilde{u} \in L^\infty(Q)$), it is deduced from (106) and (107) that

$$\begin{aligned} &\frac{d}{dt}\left(\|U\|^2 + K\|V\|^2\right) + \|\nabla U\|^2 + (K - 2C\, \|\tilde{u}\|_{L^\infty}^2)\|V\|_{H^1}^2 \\ &\qquad \leq\ C\, \alpha^4 \left(1 + \|\nabla \tilde{v}\|_{L^6}^4\right) \|U\|^2 + C\, K\, \alpha^4 \|f\|_{L^2}^4 \|V\|^2 \\ &\qquad + \alpha \Big(\|g_u\|^2 + K\|g_v\|^2 + (1+K)\|U\|^2 + K\|V\|^2 \Big). \end{aligned}$$

Using that $U(0) = V(0) = 0$ and $\|g_u\|_{L^2(L^2)}$, $\|g_v\|_{L^2(L^2)}$, $\|f\|_{L^4(L^2)}$, $\|\tilde{u}\|_{L^\infty(Q)}$, and $\|\nabla\tilde{v}\|_{L^4(L^6)}$ are constant finite values. In fact, the data are even more regular because $g_u, g_v \in L^4(Q)$, and since $\tilde{v} \in X_4$, $\tilde{u} \in L^\infty(Q)$ (thanks to (41)) and $\nabla\tilde{v} \in L^4(0,T;L^6(\Omega)$. Therefore, the Gronwall lemma implies that

$$\|(U,V)\|_{X\times X} \leq C. \tag{108}$$

Consequently, by applying the Leray–Schauder fixed-point theorem, one has the existence of $(U,V) \in X_{20/11} \times X_{20/11}$ a solution of problem (96). The uniqueness of solution is directly deduced from the linearity of problem (96).

Finally, it suffices to prove that this solution is in fact more regular; namely, $(U,V) \in X_4 \times X_4$. Indeed, from (108) and using (101) and Lemma 6, it can be deduced that

$$U,\, V \in X_{20/11} \hookrightarrow L^\infty(L^4) \cap L^{20/11}(H^{37/20}) \hookrightarrow L^{20/3}(Q), \tag{109}$$

and since $\tilde{f} \in L^4(Q_c)$, one has $\tilde{f}\, V 1_{\Omega_c} \in L^{5/2}(Q)$. Then, applying Theorem 1 (for $p = 5/2$) to $(105)_2$, it can be deduced that

$$V \in X_{5/2}.$$

By Sobolev embeddings, for $p = 5/2$, one has $W^{2-2/p,p}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q < +\infty$. Then $V \in L^q(Q)$ $(q < +\infty)$; hence, $\tilde{f}\, V 1_{\Omega_c}$ is bounded in $L^{4-\varepsilon}(Q)$, for any $\varepsilon > 0$. Thus, using (109), $U \in L^{20/3}(Q)$. Again, from Theorem 1 (for $p = 4-\varepsilon$), one deduces that $V \in X_{4-\varepsilon}$. This last regularity implies that $V \in L^\infty(Q)$, and the same argument leads to

$$V \in X_4. \tag{110}$$

Now, using that $U \in X_{20/11}$ and therefore satisfies (102) and (109) and that the regularity (110) implies that $\Delta\tilde{v} \in L^4(Q)$ and from (70) $\nabla\tilde{v} \in L^{20}(Q)$, it is obtained that

$$\nabla \cdot (U\nabla\tilde{v}) = U\,\Delta\tilde{v} + \nabla U \cdot \nabla\tilde{v} \in L^{5/2}(Q).$$

Now, using that $\Delta V \in L^4(Q)$ and $\tilde{u} \in X_4$ (hence, $\tilde{u} \in L^\infty(Q)$, $\nabla\tilde{u} \in L^{20}(Q)$, and $\Delta\tilde{u} \in L^4(Q)$),

$$\nabla \cdot (\tilde{u}\nabla V) = \tilde{u}\,\Delta V + \nabla\tilde{u} \cdot \nabla V \in L^4(Q).$$

Applying Theorem 1 (for $p = 5/2$) to $(96)_1$, it is first deduced that $U \in X_{5/2}$. Second, using the reasoning made for V above, it is deduced that $U \in L^q(Q)$ (for any $q < +\infty$). Moreover, $U \in L^\infty(W^{6/5,5/2}) \cap L^{5/2}(W^{2,5/2})$, and thus

$$\nabla U \in L^\infty(W^{1/5,5/2}) \cap L^{5/2}(W^{1,5/2}) \hookrightarrow L^\infty(L^3) \cap L^{5/2}(W^{1,5/2}) \hookrightarrow L^5(Q),$$

which implies

$$\nabla \cdot (U\nabla\tilde{v}) = U\,\Delta\tilde{v} + \nabla U \cdot \nabla\tilde{v} \in L^{4-\varepsilon}(Q) + L^4(Q) \quad \varepsilon > 0\,(\text{small}),$$

and therefore $U \in X_{4-\varepsilon}$. This last condition guarantees that $U \in L^\infty(Q)$, and thus it can easily be deduced that $U \in X_4$. ∎

Now the existence of Lagrange multiplier for problem (74) associated to any local optimal solution $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ will be shown.

THEOREM 10. *Let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ be a local optimal solution for the control problem* (74). *Then there exists a Lagrange multiplier $\xi = (\lambda, \eta) \in L^{4/3}(Q) \times L^{4/3}(Q)$ such that for all $(U, V, F) \in \widehat{X}_4 \times \widehat{X}_4 \times \mathcal{C}(\tilde{f})$*

$$\begin{aligned}&\gamma_u \int_0^T \int_\Omega \operatorname{sgn}(\tilde{u} - u_d)|\tilde{u} - u_d|^{13/7} U + \gamma_v \int_0^T \int_\Omega (\tilde{v} - v_d)V \\ &\quad + \gamma_f \int_0^T \int_{\Omega_c} (\tilde{f})^3 F - \int_0^T \int_\Omega \Big(\partial_t U - \Delta U - \nabla \cdot (U \nabla \tilde{v}) - \nabla \cdot (\tilde{u} \nabla V)\Big)\lambda \\ &\quad - \int_0^T \int_\Omega \Big(\partial_t V - \Delta V + V - U - \tilde{f} V 1_{\Omega_c}\Big)\eta + \int_0^T \int_{\Omega_c} F \tilde{v} \eta \geq 0. \end{aligned} \tag{111}$$

Proof. From Lemma 12, $\tilde{s} \in \mathcal{S}_{ad}$ is a regular point. Then, from Theorem 9, there exists a Lagrange multiplier $\xi = (\lambda, \eta) \in L^{4/3}(Q) \times L^{4/3}(Q)$ such that by $(91)_2$ and Remark 7, one must satisfy

$$\mathcal{L}'(s, \lambda, \eta)[r] = J'(\tilde{s})[r] - \langle \lambda, G_1'(\tilde{s})[r] \rangle_{L^{4/3}} - \langle \eta, G_2'(\tilde{s})[r] \rangle_{L^{4/3}} \geq 0 \tag{112}$$

for all $r = (U, V, F) \in \widehat{X}_4 \times \widehat{X}_4 \times \mathcal{C}(\tilde{f})$. Thus, the proof follows from (94), (95), and (112). ∎

From Theorem 10, an optimality system for problem (74) can be derived.

COROLLARY 3. *Let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ be a local optimal solution for the control problem* (74). *Then the Lagrange multiplier $(\lambda, \eta) \in L^{4/3}(Q) \times L^{4/3}(Q)$, provided by Theorem* 10, *satisfies the system*

$$\begin{aligned}&\int_0^T \int_\Omega \Big(\partial_t U - \Delta U - \nabla \cdot (U \nabla \tilde{v})\Big)\lambda - \int_0^T \int_\Omega U\eta \\ &\qquad = \gamma_u \int_0^T \int_\Omega \operatorname{sgn}(\tilde{u} - u_d)|\tilde{u} - u_d|^{13/7} U \qquad \forall\, U \in \widehat{X}_4, \end{aligned} \tag{113}$$

$$\begin{aligned}&\int_0^T \int_\Omega \Big(\partial_t V - \Delta V + V\Big)\eta - \int_0^T \int_{\Omega_c} \tilde{f} V \eta - \int_0^T \int_\Omega \nabla \cdot (\tilde{u} \nabla V)\lambda \\ &\qquad = \gamma_v \int_0^T \int_\Omega (\tilde{v} - v_d)V \qquad \forall\, V \in \widehat{X}_4 \end{aligned} \tag{114}$$

and the optimality condition

$$\int_0^T \int_{\Omega_c} (\gamma_f (\tilde{f})^3 + \tilde{v}\eta)(f - \tilde{f}) \geq 0 \qquad \forall f \in \mathcal{F}. \tag{115}$$

Proof. From (111), taking $(V, F) = (0, 0)$ and using that $\widehat{X}_4$ is a vectorial space, (113) holds. Similarly, taking $(U, F) = (0, 0)$ in (111) and taking into account that $\widehat{X}_4$ is a vectorial space, (114) is deduced. Finally, taking $(U, V) = (0, 0)$ in (111), it holds that

$$\gamma_f \int_0^T \int_{\Omega_c} (\tilde{f})^3 F + \int_0^T \int_{\Omega_c} \tilde{v}\eta F \geq 0, \quad \forall\, F \in \mathcal{C}(\tilde{f}).$$

Thus, choosing $F = \theta(f - \tilde{f}) \in \mathcal{C}(\tilde{f})$ for all $f \in \mathcal{F}$ and $\theta \geq 0$ in the last inequality, (115) is deduced. ∎

Remark 9. A pair $(\lambda,\eta)\in L^{4/3}(Q)\times L^{4/3}(Q)$ satisfying (113)–(114) corresponds to the concept of very weak solution of the linear system

$$\text{(116)}\quad \left\{\begin{array}{r}-\partial_t\lambda-\Delta\lambda+\nabla\lambda\cdot\nabla\tilde v-\eta=\gamma_u\mathrm{sgn}(\tilde u-u_d)|\tilde u-u_d|^{13/7}\quad \text{in } Q,\\ -\partial_t\eta-\Delta\eta-\nabla\cdot(\tilde u\nabla\lambda)+\eta-\tilde f\,\eta\,1_{\Omega_c}=\gamma_v(\tilde v-v_d)\quad \text{in } Q,\\ \lambda(T)=0,\ \eta(T)=0\quad \text{in } \Omega,\\ \dfrac{\partial\lambda}{\partial\mathbf n}=0,\ \dfrac{\partial\eta}{\partial\mathbf n}=0\quad \text{on } (0,T)\times\partial\Omega.\end{array}\right.$$

THEOREM 11. *Let $\tilde s=(\tilde u,\tilde v,\tilde f)\in\mathcal S_{ad}$ be a local optimal solution for the problem* (74) *and $u_d\in L^{26/7}(Q)$. Then the system* (116) *has a unique solution (λ,η) such that*

$$\text{(117)}\qquad \lambda\in X_{20/13},$$

$$\text{(118)}\qquad \eta\in X_{20/13}.$$

Proof. Since the desired state $u_d\in L^{20/7}(Q)$, it can be deduced that $h(\tilde u):=\mathrm{sgn}(\tilde u-u_d)|\tilde u-u_d|^{13/7}\in L^{20/13}(Q)$. Let $s=T-t$, with $t\in(0,T)$ and $\tilde\lambda(s)=\lambda(t)$, $\tilde\eta(s)=\eta(t)$. Then system (116) is equivalent to

$$\text{(119)}\qquad \left\{\begin{array}{r}\partial_s\tilde\lambda-\Delta\tilde\lambda+\nabla\tilde\lambda\cdot\nabla\tilde v-\tilde\eta=\gamma_u h(\tilde u)\quad \text{in } Q,\\ \partial_s\tilde\eta-\Delta\tilde\eta-\nabla\cdot(\tilde u\nabla\tilde\lambda)+\tilde\eta-\tilde f\,\tilde\eta\,1_{\Omega_c}=\gamma_v(\tilde v-v_d)\quad \text{in } Q,\\ \tilde\lambda(0)=0,\ \tilde\eta(0)=0\quad \text{in } \Omega,\\ \dfrac{\partial\tilde\lambda}{\partial\mathbf n}=0,\ \dfrac{\partial\tilde\eta}{\partial\mathbf n}=0\quad \text{on } (0,T)\times\partial\Omega.\end{array}\right.$$

In order to prove the existence of a solution for (119), as before the Leray–Schauder fixed-point theorem it will be applied over the operator

$$\text{(120)}\qquad \widehat T:(\bar\lambda,\bar\eta)\in X\times X\mapsto(\lambda,\eta)\in X_{20/13}\times X_{20/13},$$

where the space X is defined in (4). This time, it is wanted to prove Theorem 2 for operator $\widehat T$, using $\mathcal X=X\times X$ and $\mathcal Y=X_{20/13}\times X_{20/13}$, and $(\lambda,\eta)=\widehat T(\bar\lambda,\bar\eta)$ solving the decoupled problem

$$\text{(121)}\qquad \left\{\begin{array}{r}\partial_s\lambda-\Delta\lambda+\nabla\bar\lambda\cdot\nabla\tilde v-\bar\eta=\gamma_u h(\tilde u)\quad \text{in } Q,\\ \partial_s\eta-\Delta\eta-\nabla\cdot(\tilde u\nabla\lambda)+\eta-\tilde f\,\bar\eta\,1_{\Omega_c}=\gamma_v(\tilde v-v_d)\quad \text{in } Q,\\ \lambda(0)=0,\ \eta(0)=0\quad \text{in } \Omega,\\ \dfrac{\partial\lambda}{\partial\mathbf n}=0,\ \dfrac{\partial\eta}{\partial\mathbf n}=0\quad \text{on } (0,T)\times\partial\Omega.\end{array}\right.$$

Step 1: In order to prove Lemma 1, $(\bar\lambda,\bar\eta)\in X\times X$ is taken. Reasoning over $(121)_1$ and using Corollary 1, it can be deduced that

$$-\nabla\bar\lambda\cdot\nabla\tilde v+\bar\eta+\gamma_u h(\tilde u)\in L^{20/11}(Q)+L^{10/3}(Q)+L^{20/13}(Q),$$

which implies (by Theorem 1) that $\lambda\in X_{20/13}$. Such regularity cannot be improved due to the regularity of $\gamma_u h(\tilde u)$.

Now the reasoning focus on $(121)_2$. Observe that

$$\text{(122)}\qquad \tilde f\,\bar\eta\,1_{\Omega_c}+\gamma_v(\tilde v-v_d)\in L^{20/11}(Q)+L^2(Q).$$

From $\lambda\in X_{20/13}$, it can be deduced that

$$\text{(123)}\qquad \begin{array}{rcl}\lambda\in L^\infty(W^{7/10,20,13})\cap L^{20/13}(W^{2,20/13}) & \hookrightarrow & L^\infty(H^{1/4})\cap L^{20/13}(H^{31/20})\\ & \hookrightarrow & L^p(H^{1/4+2/p})\quad p\ge 20/13.\end{array}$$

Therefore, $\nabla\lambda \in L^p(H^{-3/4+2/p}) \hookrightarrow L^{20/9}(Q)$ and

$$\nabla\cdot(\tilde{u}\nabla\lambda) = \tilde{u}\Delta\lambda + \nabla\tilde{u}\cdot\nabla\lambda \in L^{20/13}(Q) + L^2(Q). \tag{124}$$

From (122) and (124), it is deduced that $\eta \in X_{20/13}$. Again, the regularity for η cannot be improved because the regularity of λ cannot either. In conclusion, one has that operator $\widehat{T}$ is well defined from $X \times X$ in $X_{20/13} \times X_{20/13}$ and maps bounded sets of $X \times X$ in bounded sets of $X_{20/13} \times X_{20/13}$.

Step 2: Lemma 9 guarantees that $X_{20/13}$ is compactly embedded in X. Therefore, Lemma 2 is true in this case.

Step 3: In particular, using the argument of Lemma 14, it is not difficult to prove the continuity of $\widehat{T}$ from $X \times X$ to itself.

Step 4: The aim is to show that the set

$$\widehat{T}_\alpha := \{(\lambda,\eta) \in X_{20/13} \times X_{20/13} \,:\, (\lambda,\eta) = \alpha\widehat{T}(\lambda,\eta) \text{ for some } \alpha \in [0,1]\}$$

is bounded in $X \times X$ (with respect to α). Indeed, if $(\lambda,\eta) \in \widehat{T}_\alpha$, then $(\lambda,\eta) \in X_{20/13} \times X_{20/13}$ and solves the coupled linear problem

$$\begin{cases} \partial_s\lambda - \Delta\lambda + \alpha\,\nabla\lambda\cdot\nabla\tilde{v} - \alpha\,\eta = \alpha\,\gamma_u h(\tilde{u}) \quad \text{in } Q, \\ \partial_s\eta - \Delta\eta - \nabla\cdot(\tilde{u}\nabla\lambda) + \eta - \alpha\,\tilde{f}\,\eta\,1_{\Omega_c} = \alpha\,\gamma_v(\tilde{v}-v_d) \quad \text{in } Q, \\ \lambda(0)=0,\ \eta(0)=0 \quad \text{in } \Omega, \\ \dfrac{\partial\lambda}{\partial\mathbf{n}} = 0,\ \dfrac{\partial\eta}{\partial\mathbf{n}} = 0 \quad \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{125}$$

Taking λ as test function in $(125)_1$, it is obtained that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\lambda\|^2 + \|\nabla\lambda\|^2 \\ &\le \alpha\left(\|\eta\|^2 + \|\lambda\|^2\right) + \alpha\,\|\lambda\|_{H^1}\|\nabla\tilde{v}\|_{L^6}\|\lambda\|_{L^3} + \alpha\gamma_u\,\|h(\tilde{u})\|_{L^{20/13}}\|\lambda\|_{20/7} \\ &\le \varepsilon\left(\|\nabla\lambda\|^2 + \|\lambda\|^2\right) + \alpha\left(\|\eta\|^2 + \|\lambda\|^2\right) \\ &\quad + C_\varepsilon\left(\alpha^4\,\|\nabla\tilde{v}\|_{L^6}^4\|\lambda\|^2 + \alpha^{40/31}\,\|h(\tilde{u})\|_{L^{20/13}}^{20/13}\,\|\lambda\|^2 + \alpha^{40/31}\,\|h(\tilde{u})\|_{L^{20/13}}^{15/13}\right). \end{aligned} \tag{126}$$

Taking η as test function in $(125)_2$, it is obtained that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\eta\|^2 + \|\eta\|_{H^1}^2 \\ &\le \|\tilde{u}\|_{L^\infty}\|\nabla\lambda\|\|\nabla\eta\| + \alpha\,\|\tilde{f}\|\|\eta\|_{L^6}\|\eta\|_{L^3} + \alpha\,\gamma_v\,\|\tilde{v}-v_d\|\|\eta\| \\ &\le \varepsilon\,\|\nabla\eta\|^2 + C_\varepsilon\left(\|\tilde{u}\|_{L^\infty}^2\|\nabla\lambda\|^2 + \alpha\,\|\tilde{f}\|^4\|\eta\|^2 + \alpha\left(\gamma_v^2\,\|\tilde{v}-v_d\|^2 + \|\eta\|^2\right)\right). \end{aligned} \tag{127}$$

Considering an adequate constant $K > C_\varepsilon\,\|\tilde{u}\|_{L^\infty}^2$ (recall that $\tilde{u} \in L^\infty(Q)$), it can be deduced from (126) and (127) that

$$\begin{aligned} &\frac{d}{dt}\left(\|\eta\|^2 + K\|\lambda\|^2\right) + \|\eta\|_{H^1}^2 + K\|\nabla\lambda\|^2 \\ &\le C(\alpha)\Bigg\{\Bigg(1 + \|\nabla\tilde{v}\|_{L^6}^4 \\ &\qquad + \left(1 + \|\tilde{f}\|^4\right)\,\|\eta\|^2\|h(\tilde{u})\|_{L^{20/13}}^{20/13}\Bigg)\,\|\lambda\|^2 + \|\tilde{v}-v_d\|^2 + \|h(\tilde{u})\|_{L^{20/13}}^{15/13}\Bigg\}. \end{aligned}$$

Using Remark 4, the hypotheses of the Gronwall lemma are satisfied, which implies that

$$\|(\lambda,\eta)\|_{X\times X} \le C(\|\tilde u\|_{X_4}, \|\tilde v\|_{X_4}, \|\tilde f\|_{L^4(Q)}, \|u_d\|_{L^{20/7}(Q)}, \|v_d\|_{L^2(Q)}).$$

Therefore, by applying the Leray–Schauder fixed-point theorem, the existence of a solution of problem (116), $(\lambda,\eta)\in X_{20/13}\times X_{20/13}$, is obtained. The uniqueness of a solution is directly deduced from the linearity of problem (116). ∎

In the following result, more regularity for the Lagrange multiplier (λ,η) than provided by Theorem 10 will be obtained.

THEOREM 12. *Let $\tilde s = (\tilde u, \tilde v, \tilde f) \in \mathcal{S}_{ad}$ be a local optimal solution for the control problem* (74)*. Then the Lagrange multiplier, provided by Theorem* 10*, satisfies* $(\lambda,\eta)\in X_{20/13}\times X_{20/13}$.

Proof. Let (λ,η) be the Lagrange multiplier given in Theorem 10, which is a very weak solution of problem (116). In particular, (λ,η) satisfies (113)–(114).

Furthermore, from Theorem 11, system (116) has a unique solution $(\overline{\lambda},\overline{\eta})\in X_{20/13}\times X_{20/13}$. Then it suffices to identify (λ,η) with $(\overline{\lambda},\overline{\eta})$. With this objective, the unique solution $(U,V)\in X_4\times X_4$ of linear system (96) for $g_u := \mathrm{sgn}(\lambda-\overline{\lambda})|\lambda-\overline{\lambda}|^{1/3}\in L^4(Q)$ and $g_v := \mathrm{sgn}(\eta-\overline{\eta})|\eta-\overline{\eta}|^{1/3}\in L^4(Q)$ is considered (see Lemma 12). Then, writing (116) for $(\overline{\lambda},\overline{\eta})$ (instead of (λ,η)), testing the first equation by U and the second one by V, and integrating by parts in Ω, it is obtained that

$$\int_0^T\!\!\int_\Omega \Big(\partial_t U - \Delta U - \nabla\cdot(U\nabla\tilde v)\Big)\overline{\lambda} - \int_0^T\!\!\int_\Omega U\overline{\eta} = \gamma_u\int_0^T\!\!\int_\Omega \mathrm{sgn}(\tilde u - u_d)|\tilde u-u_d|^{13/7}U, \tag{128}$$

$$\int_0^T\!\!\int_\Omega \Big(\partial_t V - \Delta V + V - \tilde f V 1_{\Omega_c}\Big)\overline{\eta} - \int_0^T\!\!\int_\Omega \nabla\cdot(\tilde u\nabla V)\overline{\lambda} = \gamma_v\int_0^T\!\!\int_\Omega (\tilde v - v_d)V. \tag{129}$$

Making the difference between (113) for (λ,η) and (128) for $(\overline{\lambda},\overline{\eta})$ and between (114) and (129) and then adding the respective equations, since the right-hand-side terms vanish, it can be deduced that

$$\begin{aligned}&\int_0^T\!\!\int_\Omega \Big(\partial_t U - \Delta U - \nabla\cdot(U\nabla\tilde v) - \nabla\cdot(\tilde u\nabla V)\Big)(\lambda-\overline{\lambda})\\ &+\int_0^T\!\!\int_\Omega \Big(\partial_t V - \Delta V + V - U - \tilde f V 1_{\Omega_c}\Big)(\eta-\overline{\eta}) = 0.\end{aligned}\tag{130}$$

Therefore, taking into account that (U,V) is the unique solution of (96) for $g_u = \mathrm{sgn}(\lambda-\overline{\lambda})|\lambda-\overline{\lambda}|^{1/3}$ and $g_v = \mathrm{sgn}(\eta-\overline{\eta})|\eta-\overline{\eta}|^{1/3}$, from (130) it is deduced that

$$\|\lambda-\overline{\lambda}\|^{4/3}_{L^{4/3}(Q)} + \|\eta-\overline{\eta}\|^{4/3}_{L^{4/3}(Q)} = 0,$$

which implies that $(\lambda,\eta) = (\overline{\lambda},\overline{\eta})$ in $L^{4/3}(Q)\times L^{4/3}(Q)$. As a consequence of the regularity of $(\overline{\lambda},\overline{\eta})$, it holds that $(\lambda,\eta)\in X_{20/13}\times X_{20/13}$. ∎

COROLLARY 4 (optimality system). *Let $\tilde s = (\tilde u,\tilde v,\tilde f)\in\mathcal{S}_{ad}$ be a local optimal solution for the control problem* (74)*. Then the Lagrange multiplier* $(\lambda,\eta)\in X_{20/13}\times$

$X_{20/13}$ *satisfies the optimality system*

(131)
$$\begin{cases} -\partial_t\lambda - \Delta\lambda + \nabla\lambda\cdot\nabla\tilde{v} - \eta = \gamma_u \mathrm{sgn}(\tilde{u}-u_d)|\tilde{u}-u_d|^{13/7} \quad a.e.\ (t,x)\in Q, \\ -\partial_t\eta - \Delta\eta - \nabla\cdot(\tilde{u}\nabla\lambda) + \eta - \tilde{f}\,\eta\,1_{\Omega_c} = \gamma_v(\tilde{v}-v_d) \quad a.e.\ (t,x)\in Q, \\ \lambda(T)=0,\ \eta(T)=0 \quad in\ \Omega, \\ \dfrac{\partial\lambda}{\partial\mathbf{n}}=0,\ \dfrac{\partial\eta}{\partial\mathbf{n}}=0 \quad on\ (0,T)\times\partial\Omega, \\ \displaystyle\int_0^T\int_{\Omega_c}(\gamma_f(\tilde{f})^3+\tilde{v}\,\eta)(f-\tilde{f})\geq 0 \quad \forall f\in\mathcal{F}. \end{cases}$$

Remark 10. If $\gamma_f > 0$ and there is no convexity constraint on the control, that is, $\mathcal{F}\equiv L^4(Q_c)$, then $(131)_5$ becomes

$$\gamma_f(\tilde{f})^3 1_{\Omega_c} + \tilde{v}\,\eta\,1_{\Omega_c} = 0.$$

Thus, the control $\tilde{f}$ is given by

$$\tilde{f} = \left(-\frac{1}{\gamma_f}\tilde{v}\,\eta\right)^{1/3} 1_{\Omega_c}.$$

Appendix A. Existence of strong solutions of Problem (16). In this appendix, Theorem 6 will be proved. By considering the *weak* space

$$X := C(L^2)\cap L^2(H^1),$$

and the operator $R : X\times X \to X_{5/3}\times X_{10/3} \hookrightarrow X\times X$ defined by $R(\overline{u}^\varepsilon,\overline{z}^\varepsilon) = (u^\varepsilon, z^\varepsilon)$, where $(u^\varepsilon, z^\varepsilon)$ is the solution of the decoupled linear problem

$$(132)\qquad \begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon = \nabla\cdot(\overline{u}^\varepsilon_+\nabla v(\overline{z}^\varepsilon)) \quad \text{in } Q, \\ \partial_t z^\varepsilon - \Delta z^\varepsilon + z^\varepsilon = \overline{u}^\varepsilon + f\,\overline{v}^\varepsilon_+ 1_{\Omega_c} \quad \text{in } Q, \\ u^\varepsilon(0)=u_0^\varepsilon,\ z^\varepsilon(0) = v_0^\varepsilon - \varepsilon\Delta v_0^\varepsilon \quad \text{in } \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial\mathbf{n}}=0,\ \dfrac{\partial z^\varepsilon}{\partial\mathbf{n}}=0 \quad \text{on } (0,T)\times\partial\Omega, \end{cases}$$

being $\overline{v}^\varepsilon := v(\overline{z}^\varepsilon)$ is the unique solution of problem (17) and $\overline{v}^\varepsilon_+$ its positive part. Then a solution of system (16) is a fixed point of R. Therefore, in order to prove the existence of solution to system (16), we will use the Leray–Schauder fixed-point theorem (Theorem 2) taking $T = R$, $\mathcal{X} = X\times X$, and $\mathcal{Y} = X_{5/3}\times X_{10/3}$.

Step 1: Now Lemma 1 is rewritten as follows.

LEMMA 13. *The operator R is well defined from $X\times X$ to $X_{5/3}\times X_{10/3}$ and maps bounded set of $X\times X$ into bounded sets of $X_{5/3}\times X_{10/3}$*

Proof. Let $(\overline{u}^\varepsilon,\overline{z}^\varepsilon)\in X\times X$. From the H^2 and H^3-regularity of problem (17) (see [16, Theorems 2.4.2.7 and 2.5.11], respectively, and recall that $\partial\Omega\in C^{2,1}$), it can be obtained that

$$\overline{v}^\varepsilon \in L^\infty(H^2)\cap L^2(H^3).$$

Thus, it has been deduced that $\nabla\overline{v}^\varepsilon \in L^\infty(H^1)\cap L^2(H^2)\hookrightarrow L^{10}(Q)$ and $\Delta\overline{v}^\varepsilon \in L^\infty(L^2)\cap L^2(H^1)\hookrightarrow L^{10/3}(Q)$. Then, taking into account that $\overline{u}^\varepsilon\in X\hookrightarrow L^{10/3}(Q)$, it can also be deduced that $\nabla\cdot(\overline{u}^\varepsilon_+\nabla\overline{v}^\varepsilon) = \overline{u}^\varepsilon_+\Delta\overline{v}^\varepsilon + \nabla\overline{u}^\varepsilon_+\cdot\nabla\overline{v}^\varepsilon \in L^{5/3}(Q)$. Then, by Theorem 1 (for $p = 5/3$), there exists a unique solution $u^\varepsilon\in X_{5/3}$ of $(132)_1$ that satisfies

$$(133)\qquad \|u^\varepsilon\|_{X_{5/3}} \leq C(\|u_0^\varepsilon\|_{W^{4/5,5/3}}, \|\overline{u}^\varepsilon\|_X, \|\overline{z}^\varepsilon\|_X).$$

Now, since $X \hookrightarrow L^{10/3}(Q)$ and $\overline{v}^\varepsilon \in L^\infty(Q)$, it follows that $\overline{u}^\varepsilon + f\,\overline{v}^\varepsilon_+ 1_{\Omega_c} \in L^{10/3}(Q)$. Then, by Theorem 1 (for $p = 10/3$), there exists a unique solution z^ε of $(132)_2$ belonging to $X_{10/3}$ and

$$\|z^\varepsilon\|_{X_{10/3}} \le C(\|z_0^\varepsilon\|_{W_{\mathbf{n}}^{7/5,10/3}}, \|\overline{u}^\varepsilon\|_X, \|\overline{z}^\varepsilon\|_X, \|f\|_{L^4(Q)}). \tag{134}$$

Therefore, R is well defined from $X \times X$ in $X_{5/3} \times X_{10/3}$ and maps bounded set of $X \times X$ into bounded sets of $X_{5/3} \times X_{10/3}$. □

Step 2: From Corollary 2, the compact embedding of $X_{10/3}$ and $X_{5/3}$ in X can be deduced. Therefore, Lemma 2 is proved in this case.

Step 3: Now Lemma 3 is rewritten as follows.

LEMMA 14. *The operator $R : X \times X \to X \times X$ is continuous.*

Proof. Let $\{(\overline{u}^\varepsilon_m, \overline{z}^\varepsilon_m)\}_{m\in\mathbb{N}} \subset X \times X$ be a sequence such that

$$(\overline{u}^\varepsilon_m, \overline{z}^\varepsilon_m) \to (\overline{u}^\varepsilon, \overline{z}^\varepsilon) \text{ in } X \times X. \tag{135}$$

In particular, $\{(\overline{u}^\varepsilon_m, \overline{z}^\varepsilon_m)\}_{m\in\mathbb{N}}$ is bounded in $X \times X$. Thus, from (133) and (134), the boundedness of the sequence $\{(u^\varepsilon_m, z^\varepsilon_m) := R(\overline{u}^\varepsilon_m, \overline{z}^\varepsilon_m)\}_{m\in\mathbb{N}}$ in $X_{5/3} \times X_{10/3}$ can be deduced. Then there exists a subsequence of $\{R(\overline{u}^\varepsilon_m, \overline{z}^\varepsilon_m)\}_{m\in\mathbb{N}}$, still denoted by $\{R(\overline{u}^\varepsilon_m, \overline{z}^\varepsilon_m)\}_{m\in\mathbb{N}}$, and an element $(\widehat{u}^\varepsilon, \widehat{z}^\varepsilon) \in X_{5/3} \times X_{10/3}$ such that

$$R(\overline{u}^\varepsilon_m, \overline{z}^\varepsilon_m) \to (\widehat{u}^\varepsilon, \widehat{z}^\varepsilon) \text{ weakly in } X_{5/3} \times X_{10/3} \text{ and strongly in } X \times X. \tag{136}$$

Now system (132) written for $(u^\varepsilon, z^\varepsilon) = R(\overline{u}^\varepsilon_m, \overline{z}^\varepsilon_m)$ and $(\overline{u}^\varepsilon, \overline{z}^\varepsilon) = (\overline{u}^\varepsilon_m, \overline{z}^\varepsilon_m)$ is considered. From (135) and (136), taking the limit in the system depending on m, as m goes to $+\infty$, it is deduced that $(\widehat{u}^\varepsilon, \widehat{z}^\varepsilon) = R(\lim_{m\to+\infty}(\overline{u}^\varepsilon_m, \overline{z}^\varepsilon_m))$. Then by the uniqueness of the limit, the whole sequence $\{R(\overline{u}^\varepsilon_m, \overline{z}^\varepsilon_m)\}_{m\in\mathbb{N}}$ converges to $(\widehat{u}^\varepsilon, \widehat{z}^\varepsilon)$ strongly in $X \times X$. Thus, operator $R : X \times X \to X \times X$ is continuous. □

Step 4: The proof of the boundedness of the set $\{x \in \mathcal{X} : x = \alpha\, Rx \quad \text{for some } 0 \le \alpha \le 1\}$ in this case follows from the following result.

LEMMA 15. *Let $(u_0^\varepsilon, v_0^\varepsilon - \varepsilon\Delta v_0^\varepsilon) \in W^{4/5,5/3}(\Omega) \times W_{\mathbf{n}}^{7/5,10/3}(\Omega)$ with $u_0^\varepsilon \ge 0$ in Ω and $f \in L^4(Q_c)$. Then the possible fixed points $(u^\varepsilon, v^\varepsilon)$ of αR are bounded in $X \times X$, independently of $\alpha \in [0,1]$, with $u^\varepsilon \ge 0$.*

Proof. Assume that $\alpha \in (0,1]$ (the case $\alpha = 0$ is trivial). Notice that if $(u^\varepsilon, z^\varepsilon)$ is a fixed point of $\alpha R(u^\varepsilon, z^\varepsilon)$, then $(u^\varepsilon, z^\varepsilon) \in X_{5/3} \times X_{10/3}$ and satisfies a.e. in Q the following problem:

$$\begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon = \alpha\, \nabla \cdot (u^\varepsilon_+ \nabla v^\varepsilon) \quad \text{in } Q, \\ \partial_t z^\varepsilon - \Delta z^\varepsilon + z^\varepsilon = \alpha\, u^\varepsilon + \alpha\, f\, v^\varepsilon_+ 1_{\Omega_c} \quad \text{in } Q, \\ u^\varepsilon(0) = u_0^\varepsilon,\ z^\varepsilon(0) = v_0^\varepsilon - \varepsilon\Delta v_0^\varepsilon \quad \text{in } \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial \mathbf{n}} = 0,\ \dfrac{\partial z^\varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on } (0,T) \times \partial\Omega. \end{cases} \tag{137}$$

The proof is carried out in three steps.

Step 1: u^ε is bounded in $L^\infty(L^1)$. In fact, one has

$$u^\varepsilon \ge 0 \quad \text{a.e. in } Q \quad \text{and} \quad \int_\Omega u^\varepsilon(t) = \int_\Omega u_0^\varepsilon \quad \forall\, t > 0. \tag{138}$$

Let $(u^\varepsilon, z^\varepsilon)$ be a solution of (137). Then $(u^\varepsilon, z^\varepsilon) \in X_{5/3} \times X_{10/3}$. In particular, $\partial_t u^\varepsilon$, Δv^ε, and $\nabla \cdot (u^\varepsilon_+ \nabla v^\varepsilon)$ belong to $L^{5/3}(Q)$. Testing $(137)_1$ by $u^\varepsilon_- \in X \hookrightarrow$

$L^{10/3}(Q) \hookrightarrow L^{5/2}(Q)$, where $u^\varepsilon_- := \min\{u^\varepsilon, 0\} \leq 0$, and taking into account that $u^\varepsilon_- = 0$ if $u^\varepsilon \geq 0$, $\nabla u^\varepsilon_- = \nabla u^\varepsilon$ if $u^\varepsilon \leq 0$, and $\nabla u^\varepsilon_- = 0$ if $u^\varepsilon > 0$, it can be obtained that

$$\frac{1}{2}\frac{d}{dt}\|u^\varepsilon_-\|^2 + \|\nabla u^\varepsilon_-\|^2 = -\alpha(u^\varepsilon_+ \nabla v^\varepsilon, \nabla u^\varepsilon_-) = 0,$$

which, jointly with $u^\varepsilon_0 \geq 0$ a.e. in Ω, implies that $u^\varepsilon_- \equiv 0$; hence, $u^\varepsilon \geq 0$ a.e. in Q and $u^\varepsilon_+ \equiv u^\varepsilon$ (the fact of taking u^ε_+ in the chemotactic term in $(137)_1$ is used here to guarantee the positivity of u^ε). Finally, integrating $(137)_1$ in Ω, $\int_\Omega u^\varepsilon(t) = \int_\Omega u^\varepsilon_0 := m^\varepsilon_0$ holds.

Step 2: z^ε is bounded in X.

It can be observed that $u^\varepsilon + 1 \geq 1$ and $u^\varepsilon + 1 \in L^\infty(L^1)$. Then, in particular, $u^\varepsilon + 1 \in L^1(Q)$ and

$$\frac{2}{5}\ln(u^\varepsilon + 1) = \ln\left[(u^\varepsilon + 1)^{2/5}\right] \leq (u^\varepsilon + 1)^{2/5} \in L^{5/2}(Q);$$

hence, $\ln(u^\varepsilon + 1) \in L^{5/2}(Q)$. Note that an extension of this argument yields to

$$\int_\Omega (u^\varepsilon + 1)\ln(u^\varepsilon + 1) \leq C(\|u^\varepsilon\|_{L^p}) \quad \text{for any } p > 1.$$

Now, testing $(137)_1$ by $\ln(u^\varepsilon + 1) \in L^{5/2}(Q)$ and $(137)_2$ (rewritten in terms of v^ε) by $-\Delta v^\varepsilon \in L^{10/3}(W^{2,10/3})$, it can be obtained that

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega (u^\varepsilon + 1)\ln(u^\varepsilon + 1) + \frac{1}{2}\|\nabla v^\varepsilon\|^2 + \frac{\varepsilon}{2}\|\Delta v^\varepsilon\|^2\right) + 4\|\nabla\sqrt{u^\varepsilon + 1}\|^2 \\ &\quad + \|\Delta v^\varepsilon\|^2 + \|\nabla v^\varepsilon\|^2 + \varepsilon\|\Delta v^\varepsilon\|^2 + \varepsilon\|\nabla(\Delta v^\varepsilon)\|^2 \\ &= -\alpha\int_\Omega \frac{u^\varepsilon}{u^\varepsilon + 1}\nabla v^\varepsilon \cdot \nabla u^\varepsilon + \alpha\int_\Omega \nabla u^\varepsilon \cdot \nabla v^\varepsilon - \alpha\int_\Omega f\, v^\varepsilon_+ 1_{\Omega_c}\Delta v^\varepsilon \\ &= \alpha\int_\Omega \frac{1}{u^\varepsilon + 1}\nabla u^\varepsilon \cdot \nabla v^\varepsilon - \alpha\int_\Omega f\, v^\varepsilon_+ 1_{\Omega_c}\Delta v^\varepsilon. \end{aligned} \tag{139}$$

Applying Hölder and Young inequalities, the following inequalities hold:

$$\alpha\int_\Omega \frac{1}{u^\varepsilon + 1}\nabla u^\varepsilon \cdot \nabla v^\varepsilon \leq \frac{\alpha}{2}\int_\Omega \frac{|\nabla u^\varepsilon|^2}{u^\varepsilon + 1} + \frac{\alpha}{2}\int_\Omega \frac{|\nabla v^\varepsilon|^2}{u^\varepsilon + 1} \leq 2\alpha\|\nabla\sqrt{u^\varepsilon + 1}\|^2 + \frac{\alpha}{2}\|\nabla v^\varepsilon\|^2, \tag{140}$$

$$-\alpha\int_\Omega f\, v^\varepsilon_+ 1_{\Omega_c}\Delta v^\varepsilon \leq \alpha\|f\|_{L^4}\|v^\varepsilon\|_{L^4}\|\Delta v^\varepsilon\| \leq \delta\|v^\varepsilon\|^2_{H^2} + \alpha^2 C_\delta\|f\|^2_{L^4}\|v^\varepsilon\|^2_{H^1}. \tag{141}$$

Moreover, integrating $(137)_2$ in Ω, using (138), and taking into account that v^ε is the unique solution of the problem (17), it holds that

$$\frac{d}{dt}\left(\int_\Omega v^\varepsilon\right) + \int_\Omega v^\varepsilon = \alpha\int_\Omega u^\varepsilon_0 + \alpha\int_\Omega f\, v^\varepsilon_+ 1_{\Omega_c}.$$

Multiplying this equation by $\int_\Omega v^\varepsilon$ and using the Hölder and Young inequalities, one has

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\int_\Omega v^\varepsilon\right)^2 + \left(\int_\Omega v^\varepsilon\right)^2 &= \alpha\left(\int_\Omega u^\varepsilon_0\right)\left(\int_\Omega v^\varepsilon\right) + \alpha\left(\int_\Omega f\, v^\varepsilon_+ 1_{\Omega_c}\right)\left(\int_\Omega v^\varepsilon\right) \\ &\leq \frac{1}{2}\left(\int_\Omega v^\varepsilon\right)^2 + \alpha^2 C\left(\int_\Omega u^\varepsilon_0\right)^2 + \alpha^2 C\|f\|^2\|v^\varepsilon\|^2. \end{aligned} \tag{142}$$

Adding (142) to (139) and then replacing (140) and (141) in the resulting inequality and taking into account that $\alpha \le 1$, it can be deduced that

$$\frac{d}{dt}\left(\int_\Omega (u^\varepsilon+1)\ln(u^\varepsilon+1) + \frac{1}{2}\|v^\varepsilon\|^2_{H^1} + \frac{\varepsilon}{2}\|\Delta v^\varepsilon\|^2\right) + 2\|\nabla\sqrt{u^\varepsilon+1}\|^2$$

$$+ C\|v^\varepsilon\|^2_{H^2} + \varepsilon\|\nabla(\Delta v^\varepsilon)\|^2 \le C\left(\left(\int_\Omega u_0^\varepsilon\right)^2 + \|f\|^2_{L^4}\|v^\varepsilon\|^2_{H^1}\right). \tag{143}$$

From (143) and the Gronwall lemma, taking into account $\int_\Omega (u^\varepsilon+1)\ln(u^\varepsilon+1) \le C\|u^\varepsilon+1\|_{L^p}$, for any $p>1$, it holds that

$$\|v^\varepsilon\|^2_{L^\infty(0,T;H^2(\Omega))} \le \frac{1}{\varepsilon}\exp\left(C\int_0^T \|f(s)\|^2_{L^4}ds\right)\left(\|u_0^\varepsilon\|^2_{L^{p_0}} + \|v_0^\varepsilon\|^2_{H^2} + C\left(\int_\Omega u_0^\varepsilon\right)^2 T\right)$$

$$:= K_0^\varepsilon\left(T, \|u_0^\varepsilon\|_{L^{p_0}}, \|v_0^\varepsilon\|_{H^2}, \|f\|_{L^2(L^4)}\right). \tag{144}$$

Now, integrating (143) in (0,T) and using (144), it follows that

$$\int_0^T \|v^\varepsilon(s)\|^2_{H^3}ds$$

$$\le \frac{1}{\varepsilon}C\left(\|u_0^\varepsilon\|^2_{L^{p_0}} + \|v_0^\varepsilon\|^2_{H^2} + \left(\int_\Omega u_0^\varepsilon\right)^2 T + \left(\sup_{0\le s\le T}\|v^\varepsilon(s)\|^2_{H^2}\right)\|f\|^2_{L^2(L^4)}\right)$$

$$:= K_1^\varepsilon(T, \|u_0^\varepsilon\|_{L^{p_0}}, \|v_0^\varepsilon\|_{H^2}, \|f\|_{L^2(L^4)}). \tag{145}$$

Therefore, from (144) and (145), the boundedness of v^ε in $L^\infty(H^2)\cap L^2(H^3)$ can be deduced (independently of $\alpha\in(0,1]$), which implies that z^ε is bounded in X.

Step 3: u^ε is bounded in X.

Testing $(137)_1$ by u^ε, one has

$$\frac{1}{2}\frac{d}{dt}\|u^\varepsilon\|^2 + \|\nabla u^\varepsilon\|^2 \le \alpha\|u^\varepsilon\|_{L^3}\|\nabla v^\varepsilon\|_{L^6}\|\nabla u^\varepsilon\|. \tag{146}$$

Since v^ε is bounded in $L^\infty(H^2)$, in particular ∇v^ε is bounded in $L^\infty(L^6)$, by applying (7) and Young inequalities and adding $\|u^\varepsilon\|^2$ to both sides of (146), the following inequality holds:

$$\frac{d}{dt}\|u^\varepsilon\|^2 + \|u^\varepsilon\|^2_{H^1} \le C\,\|u^\varepsilon\|^2. \tag{147}$$

Then the Gronwall lemma can be applied in (147), obtaining that u^ε is bounded in $\mathcal{X}$. Consequently, the fixed points of αR are bounded in $X\times X$, independently of $\alpha>0$. ∎

Finally, from Lemmas 13, 14, and 15, one follows that the operator R satisfies the hypotheses of the Leray–Schauder fixed-point theorem (Theorem 2). Thus, it is concluded that the map R has a fixed point $(u^\varepsilon, z^\varepsilon)\in X_{5/3}\times X_{10/3}$, that is, $R(u^\varepsilon,z^\varepsilon)=(u^\varepsilon,z^\varepsilon)$, which is a (strong) solution of system (16).

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