Logics for Order of Magnitude Qualitative Reasoning

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Outline

Qualitative Reasoning

- Qualitative reasoning is an emergent area of AI.
- It is an adequate tool for dealing with situations in which information is not sufficiently precise (e.g., numerical values are not available).
- A form of qualitative reasoning is to manage numerical data in terms of orders of magnitude [Raiman, Dague, Travé-Massuyès, . . .].

Using Logic in Qualitative Reasoning

Different approaches have been proposed which use logic in QR and study the soundness of the reasoning supported by the formalism, together with the efficiency of its use.

- Region Connection Calculus for managing qualitative spatial reasoning [Bennett, Randel et al.]
- Multimodal logics to deal with qualitative spatio-temporal representations [Bennett et al., Wolter et al.]
- Branching temporal logics to describe the possible solutions of ordinary differential equations [Shults et al.].
- Multimodal logics and PDL dealing with OMR [Burrieza, Muñoz-Velasco and Ojeda] on the basis of qualitative classes obtained from the real line divided in intervals.

Order of Magnitude Reasoning

A form of qualitative reasoning is to manage numerical data in terms of orders of magnitude.
[Raiman, Dague, Travé-Massuyès, . . .].

- Absolute Order of Magnitude (AOM), which is represented by a partition of the real line $\mathbb{R}$ and each element of $\mathbb{R}$ belongs to a qualitative class.
- Relative Order of Magnitude (ROM), introducing a family of binary order of magnitude relations which establish different comparison relations in $\mathbb{R}$ (e.g., *comparability*, *negligibility* and *closeness*).

Outline

Main results with multimodal logic

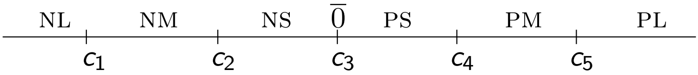
- Introduction of a logic to deal with non-closeness, distance and negligibility defined in an arbitrarily chosen strict linearly ordered set.
- The logic introduced is a special type of hybrid logic which uses a finite number of constants to represent not only points but also distances.
- We give a sound and complete axiomatization.

The qualitative classes

We consider the real line $\mathbb{R}$ divided into seven equivalence classes using five landmarks $c_i \in \mathbb{R}$, $c_j < c_{j+1}$, for all $j \in \{1, 2, 3, 4\}$:

$$\mathrm{NL} = (-\infty, c_1), \quad \mathrm{NM} = [c_1, c_2) \quad \mathrm{NS} = [c_2, c_3), \quad \overline{0} = \{c_3\}$$

$$\mathrm{PS} = (c_3, c_4], \quad \mathrm{PM} = (c_4, c_5], \quad \mathrm{PL} = (c_5, +\infty)$$



Constant distance

Definition

Let $(\mathbb{S}, <)$ a strict linearly ordered set which contains the constants c_i for $i \in \{1, 2, 3, 4, 5\}$ as defined above. Given $n \in \mathbb{N}$, we define $\overrightarrow{d_\alpha}$ as a relation in $\mathbb{S}$ such that, for every $x, y, z, x', y' \in \mathbb{S}$:

- $c_j \overrightarrow{d_\alpha} c_{j+1}$, for $j \in \{1, 2, 3, 4\}$
- If $x \overrightarrow{d_\alpha} y$, then $x < y$
- If $x \overrightarrow{d_\alpha} y$ and $x \overrightarrow{d_\alpha} z$, then $y = z$.
- If $x \overrightarrow{d_\alpha} y$, $x' \overrightarrow{d_\alpha} y'$ and $x < x'$ then $y < y'$.

Non-closeness and Distance

Definition (Non-Closeness and Distance)

Let $(\mathbb{S}, <)$ and $n \in \mathbb{N}$ be given as above. We define the relations $\overrightarrow{\mathrm{NC}}$ and $\overrightarrow{\mathrm{D}}$ in $\mathbb{S}$ as follows:

- $x\overrightarrow{\mathrm{NC}}\,y$ if and only if either $x\overline{\mathrm{OM}}y$ and $x < y$
  or there exists $z \in \mathbb{S}$ such that $z < y$ and $x\overrightarrow{d_\alpha}z$
- $x\,\overrightarrow{\mathrm{D}}\,y$ if and only if there exists $z \in \mathbb{S}$
  such that $z < y$ and $x\overrightarrow{d_\alpha}^2z$.

Negligibility

Definition (Negligibility)

Let $(\mathbb{S}, <)$ be defined as above. If $x, y \in \mathbb{S}$, we say that x is *negligible* with respect to (wrt from now on) y, usually denoted $x \overrightarrow{N} y$, if and only if, we have one of the following cases:

(i) $x = c_0$ (ii) $x \in \mathrm{NS} \cup \mathrm{PS}$ and, either $c_2 \overleftarrow{\mathrm{D}} y$ or $c_4 \overrightarrow{\mathrm{D}} y$

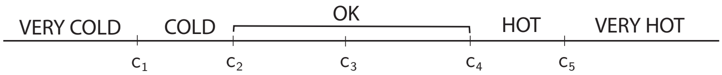
The modal connectives

Intuitive meanings

- $\overrightarrow{\Box} A$ means *A is true for all point greater than the current one.*
- $\Box_{\overrightarrow{d_\alpha}} A$ is read *A is true for all point which is greater than the current one and its distance to this one is* α.
- $\Box_{\overrightarrow{\mathrm{N}}} A$ is read *A is true for all point with respect to which the current one is negligible.*
- $\Box_{\overrightarrow{\mathrm{NC}}} A$ is read *A is true for all point which is non-close and greater than the current one.*
- $\Box_{\overrightarrow{\mathrm{D}}} A$ is read *A is true for all point which is distant from and greater than the current one.*

An example

A device to automatically control temperature



The proper axioms

- OK $\longrightarrow$ *off*
- HOT $\longrightarrow$ *cooling*
- VERY_HOT $\longrightarrow$ *X-cooling*
- COLD $\longrightarrow$ *heating*
- VERY_COLD $\longrightarrow$ *X-heating*
- (COLD $\vee$ HOT) $\longrightarrow$ *humidifier*
- (VERY_COLD $\vee$ VERY_HOT) $\longrightarrow$ *X-humidifier*

An example

Some consequences using our logic

1. $cooling \longrightarrow \overrightarrow{\Box}(\neg X\text{-}cooling \longrightarrow humidifier)$
2. $(off \wedge \neg \overline{c}_0) \longrightarrow \Box_{\overrightarrow{\mathrm{N}}} X\text{-}humidifier$
3. $(\mathrm{OK} \wedge \overleftarrow{\Diamond}\, \overline{c}_0) \longrightarrow$
   $(\Box_{\overrightarrow{\mathrm{NC}}}(humidifier \vee X\text{-}humidifier) \wedge \Box_{\overrightarrow{\mathrm{D}}} X\text{-}humidifier)$
4. $\overline{c}_1 \longrightarrow (\Box_{\mathrm{NC}}\, non\text{-}efficient \wedge \Box_{\mathrm{D}}\, warning)$

Axiom schemata

For modal white connectives and for constants

K1 $\overrightarrow{\Box}(A \longrightarrow B) \longrightarrow (\overrightarrow{\Box} A \longrightarrow \overrightarrow{\Box} B)$

K2 $A \longrightarrow \overrightarrow{\Box}\overleftarrow{\Diamond} A$

K3 $\overrightarrow{\Box} A \longrightarrow \overrightarrow{\Box}\overrightarrow{\Box} A$

K4 $\left(\overrightarrow{\Box}(A \vee B) \wedge \overrightarrow{\Box}(\overrightarrow{\Box} A \vee B) \wedge \overrightarrow{\Box}(A \vee \overrightarrow{\Box} B)\right) \to (\overrightarrow{\Box} A \vee \overrightarrow{\Box} B)$

C1 $\overleftarrow{\Diamond}\,\overline{c_i} \vee \overline{c_i} \vee \overrightarrow{\Diamond}\,\overline{c_i}$, where $i \in \{1, 2, 3, 4, 5\}$

C2 $\overline{c_i} \longrightarrow (\overleftarrow{\Box} \neg \overline{c_i} \wedge \overrightarrow{\Box} \neg \overline{c_i})$, being $i \in \{1, 2, 3, 4, 5\}$

Axiom schemata

For specific modal connectives

d1 $\Box_{\overrightarrow{d\alpha}}(A \longrightarrow B) \longrightarrow (\Box_{\overrightarrow{d\alpha}} A \longrightarrow \Box_{\overrightarrow{d\alpha}} B)$

d2 $A \longrightarrow \Box_{\overrightarrow{d\alpha}} \Diamond_{\overleftarrow{d\alpha}} A.$

d3 $\overline{c}_j \longrightarrow \Diamond_{\overrightarrow{d\alpha}} \overline{c}_{j+1}$, where $j \in \{1, 2, 3, 4\}$

d4 $\overrightarrow{\Box} A \longrightarrow \Box_{\overrightarrow{d\alpha}} A$

d5 $\Diamond_{\overrightarrow{d\alpha}} A \longrightarrow \Box_{\overrightarrow{d\alpha}} A$

d6 $(\Diamond_{\overrightarrow{d\alpha}} A \wedge \overrightarrow{\Diamond} \Diamond_{\overrightarrow{d\alpha}} B) \longrightarrow \overrightarrow{\Diamond}(A \wedge \overrightarrow{\Diamond} B)$

We also consider as axioms the corresponding mirror images of K1–K4 and d1–d6.

Soundness and Completeness

Theorem

- *Every theorem of* OM^{NCD} *is a valid formula of* $\mathcal{L}(OM)^{\mathrm{NCD}}$.
- *Every valid formula of* $\mathcal{L}(OM)^{\mathrm{NCD}}$ *is a theorem of* OM^{NCD} .

Outline

The relational proof system

- Founded on the Rasiowa-Sikorski system (RS) for FOL, extended by [Golińska-Pilarek and Orłowska 05]
- Construction of the system:
  - We give a relational logic appropriate.
  - We define a validity preserving translation.
  - We construct a sound and complete relational proof system.

Some advantages of relational methods

- clear method of generating rules,
- deduction system well suited for automated purposes,
- powerful tools for performing the major reasoning tasks,
- simple way of proving completeness
- almost automatic way of transforming a complete RS system into a complete Gentzen sequent calculus system.

Main contributions

- We present a relational proof system for the relational logic associated to a logic for order of magnitude reasoning.
- We study soundness and completeness of the proof system.
- We show how it can be used for verification of validity of formulas of the logic on the basis of an example.

The relational formulas

A *relational formula* is a expression of the form xRy, where x, y are object variables or object constants, where:

- $\mathbb{OV} = \{x, y, z, \dots\}$ is the set of object variables,
- $\mathbb{OC} = \{c_1, \dots, c_5\}$ is the set of of object constants,

and R is a *relational term*. The set of relational terms is the smallest set of expressions that includes $\mathbb{RA} = \mathbb{RV} \cup \mathbb{RC}$ and is closed with respect to the operation symbols from $\mathbb{OP}$, being:

- $\mathbb{RV} = \{R, S, \dots\}$ the set of relational variables,
- A set $\mathbb{RC} = \{1, 1', <, d_\alpha\} \cup \{\Psi_1, \dots, \Psi_5\}$ of rel. constants,
- A set $\mathbb{OP} = \{-, \cup, \cap, ;, ^{-1}\}$ of operation symbols.

The validity preserving funtion

$$\tau(p) = \tau'(p); 1, \text{ for any propositional variable } p \in \mathbb{V};$$
$$\tau(c_i) = \Psi_i; 1, \text{ for any } i \in \{1, \ldots, 5\};$$
$$\tau(\neg\varphi) = -\tau(\varphi);$$
$$\tau(\varphi \vee \psi) = \tau(\varphi) \cup \tau(\psi);$$
$$\tau(\varphi \wedge \psi) = \tau(\varphi) \cap \tau(\psi);$$
$$\tau(\varphi \rightarrow \psi) = -\tau(\varphi) \cup \tau(\psi);$$

and for $\mathcal{R} \in \{<, d_\alpha\}$:

$$\tau(\Box_{\overrightarrow{\mathcal{R}}}\varphi) = -(\mathcal{R}; -\tau(\varphi));$$
$$\tau(\Box_{\overleftarrow{\mathcal{R}}}\varphi) = -(\mathcal{R}^{-1}; -\tau(\varphi)).$$

The translation τ is defined so that it preserves validity of formulas.

The rules

- The rules have the following general form:

$$\frac{\Phi}{\Phi_1 \mid \ldots \mid \Phi_n}$$

- Φ is called the *premise* of the rule and $\Phi_1, \ldots, \Phi_n$ are called its *conclusions*.
- A rule is said to be *applicable* to a set X of formulas whenever $\Phi \subseteq X$.
- An object variable in a rule is *new* whenever it appears in a conclusion of the rule and does not appear in its premise.

Decomposition rules

Let $x, y, \in \mathbb{OS}$, $R, S \in \mathbb{RT}$, z any object symbol and w a new object variable:

$$(\cup)\ \frac{x(R \cup S)y}{xRy, xSy} \qquad (-\cup)\ \frac{x{-}(R \cup S)y}{x{-}Ry \mid x{-}Sy}$$

$$(\cap)\ \frac{x(R \cap S)y}{xRy|xSy} \qquad (-\cap)\ \frac{x{-}(R \cap S)y}{x{-}Ry, x{-}Sy}$$

$$(;)\ \frac{x(R;S)y}{xRz, x(R;S)y|zSy, x(R;S)y} \qquad (-;)\ \frac{x{-}(R;S)y}{x{-}Rw, w{-}Sy}$$

$$(-)\ \frac{x{-}{-}Ry}{xRy} \qquad (^{-1})\ \frac{xR^{-1}y}{yRx} \qquad (-^{-1})\ \frac{x{-}R^{-1}y}{y{-}Rx}$$

Specific rules and axiomatic sets

(1′1) $\dfrac{xPy}{xPz, xPy \mid y1'z, xPy}$

(1′2) $\dfrac{xPy}{x1'z, xPy \mid zPy, xPy}$

(Irref<) $\dfrac{}{x < x}$

(Tran<) $\dfrac{x < y}{x < y, x < z \mid x < y, z < y}$

(C_i1) $\dfrac{}{x\Psi_i y \mid x{-}\Psi_i y}$

(C_i2) $\dfrac{x\Psi_i y}{x\Psi_i y, x1'c_i}$

(C_i3) $\dfrac{x{-}\Psi_i y}{x{-}\Psi_i y, x{-}1'c_i}$

(D1) $\dfrac{x < y}{x\, d_\alpha\, y, x < y}$

(D2) $\dfrac{x1'y}{z\, d_\alpha\, x, x1'y \mid z\, d_\alpha\, y, x1'y}$

(D3) $\dfrac{x < y}{z\, d_\alpha\, x, x < y \mid t\, d_\alpha\, y, x < y \mid z < t, x < y}$

(Ax1) $\{x1'x\}$;

(Ax2) $\{x1y\}$

(Ax3) $\{xPy, x{-}Py\}$;

(Ax4) $\{c_j\, d_\alpha\, c_{j+1}\}$, for any $j \in \{1, \dots, 4\}$

(Ax5) $\{x < y, y < x, x1'y\}$.

An example: Proving validity of $\Diamond_{\overrightarrow{d_\alpha}} p \longrightarrow \Box_{\overrightarrow{d_\alpha}} p$ 1/2

$$\underline{x -(d_\alpha ; (P;1)) \cup -(d_\alpha ; -(P;1)) y}$$

$(\cup)$

$$\underline{x -(d_\alpha ; (P;1)) y}, \underline{x -(d_\alpha ; -(P;1)) y}$$

$(-;) \times 2$ and $(-)$, w_1, w_2 new variables

$$x - d_\alpha w_1,\ \underline{w_1 -(P;1) y},\ x - d_\alpha w_2,\ w_2 (P;1) y$$

$(-;)$, w_3 new variable

$$x - d_\alpha w_1,\ w_1 - P w_3,\ w_3 - 1 y,\ x - d_\alpha w_2,\ \underline{w_2 (P;1) y}$$

$\ldots$

An example: Proving validity of $\Diamond_{\overrightarrow{d\alpha}}\, p \longrightarrow \Box_{\overrightarrow{d\alpha}}\, p$ 2/2

$$x - d_\alpha w_1,\ w_1 - P w_3,\ w_3 - 1 y,\ x - d_\alpha w_2,\ \underline{w_2 (P;1) y}$$

$(;)$ with w_3

$w_3 1 y, \ldots$ closed

$$x - d_\alpha w_1, w_1 - P w_3, x - d_\alpha w_2, \underline{w_2 P w_3}, \ldots$$

$\ldots$

$w_1 P w_3,\ w_1 - P w_3, \ldots$ closed

$$x - d_\alpha w_1, x - d_\alpha w_2, \underline{w_2 1' w_1}, \ldots$$

(D2) with x

$x - d_\alpha w_2, x\, d_\alpha w_2, \ldots$ closed

$x - d_\alpha w_1, x\, d_\alpha w_1, \ldots$ closed

Outline

Introducing Propositional Dynamic Logic

. . . and showing its advantages

- We present a Propositional Dynamic Logic with constants representing different qualitative classes
  [Harel, Blackburn, Van Benthem, Areces, TenCate]
- We exploit the possibility of constructing complex relations from simpler ones for defining the concepts of closeness and distance, together with other programming commands such as *while* . . . *do* and *repeat* . . . *until*
- We employ its theoretical support in order to study the decidability of the satisfiability problem and to show the completeness of the system.

Some formulas for qualitative arithmetic

- $\langle +_{ps} \rangle \varphi$ is true in u iff there exists u', obtained by adding a positive small number to u, such that φ is true in u'.
- $\langle nl? \rangle \varphi$ is true in u iff u is a negative large number and φ is true in u.
- $\langle +^*_{ps} \rangle \varphi$ is true in u iff there exists u', obtained by adding a finitely many small positive numbers to u, such that φ is true in u'.
- $[+_{ps} \cup +_{nm}]\varphi$ is true in u iff for every u', obtained by adding either a positive small number or a negative medium number to u, φ is true in u'.

Closeness and Distance

$$[\mathrm{c}]\,\varphi = [+_{\mathrm{ns}} \cup +_0 \cup +_{\mathrm{ps}}]\varphi$$

$[\mathrm{c}]\,\varphi$ is true in u iff for every u' which is *close to* u, i.e obtained by adding a small number to u, φ is true in u'.

$$[\mathrm{d}]\,\varphi = [+_{\mathrm{nl}} \cup +_{\mathrm{pl}}]\varphi$$

$[\mathrm{d}]\,\varphi$ is true in u iff for every u' which is *distant from* u, i.e obtained by adding a large number to u, φ is true in u'.

Returning to the device to control the temperature

- The qualitative classes $\mathrm{NL}, \mathrm{NM}, \mathrm{NS} \cup \overline{0} \cup \mathrm{PS}, \mathrm{PM}$ and PL are interpreted by the formulas: $\textsc{Very\_Cold}$, $\textsc{Cold}$, $\textsc{Ok}$, $\textsc{Hot}$ and $\textsc{Very\_Hot}$, respectively.
- Program $+_0$ means that the system is *off*; moreover $+_{\mathsf{ps}} \cup +_{\mathsf{pm}}$ and $+_{\mathsf{pl}}$, mean that a system for *heating* and *extra heating* are operating, respectively.
- Similarly we consider the meanings of programs $+_{\mathsf{nm}} \cup +_{\mathsf{ns}}$ and $+_{\mathsf{nl}}$ for *cooling* and *extra cooling* operations, respectively.

Returning to the device to control the temperature

Some valid formulas

1. $\textsc{Hot} \longrightarrow ([+_{\mathsf{pl}}]\textsc{Very\_Hot} \wedge \langle(+_{\mathsf{nm}} \cup +_{\mathsf{ns}})^*\rangle\textsc{Ok})$
2. *while... do*
   $[(\neg\textsc{Ok}?; +_{\mathsf{Sys}})^*; \textsc{Ok}?]\textsc{Ok}$
   $+_{\mathsf{Sys}} = +_{\mathsf{nl}} \cup +_{\mathsf{nm}} \cup +_{\mathsf{ns}} \cup +_{\mathsf{ps}} \cup +_{\mathsf{pm}} \cup +_{\mathsf{pl}}$
3. *repeat... until*
   $\textsc{Very\_Hot} \longrightarrow [(+_{\mathsf{nl}}; (\neg\textsc{Ok}?; +_{\mathsf{nl}})^*; \textsc{Ok}?]\textsc{Ok}$
4. $0 \longrightarrow [\mathsf{c}]\,\textsc{Ok}$
5. $\textsc{Ok} \longrightarrow [\mathsf{d}]\,(\textsc{Very\_Cold} \vee \textsc{Cold} \vee \textsc{Hot} \vee \textsc{Very\_Hot})$
6. $\textsc{Ok} \longrightarrow ([\mathsf{c}]\,\textit{efficient} \wedge [\mathsf{d}]\,\textit{warning})$

Axiom system

Axiom schemata for qualitative classes

QE $\mathsf{nl} \vee \mathsf{nm} \vee \mathsf{ns} \vee 0 \vee \mathsf{ps} \vee \mathsf{pm} \vee \mathsf{pl}$

QU $\star \longrightarrow \neg \# \text{ for every } \star \in \mathbb{C} \text{ and } \# \in \mathbb{C} - \{\star\}$

QO1 $\mathsf{nl} \longrightarrow \langle +^*_{\mathsf{ps}} \rangle\, \mathsf{nm}$

QO2 $\mathsf{nm} \longrightarrow \langle +^*_{\mathsf{ps}} \rangle\, \mathsf{ns}$

QO3 $\mathsf{ns} \longrightarrow \langle +^*_{\mathsf{ps}} \rangle\, 0$

QO4 $0 \longrightarrow \langle +^*_{\mathsf{ps}} \rangle\, \mathsf{ps}$

QO5 $\mathsf{ps} \longrightarrow \langle +^*_{\mathsf{ps}} \rangle\, \mathsf{pm}$

QO6 $\mathsf{pm} \longrightarrow \langle +^*_{\mathsf{ps}} \rangle\, \mathsf{pl}$

Axiom system

Axiom schemata for specific programs

PS1 $\mathsf{nl} \longrightarrow [+_{\mathsf{ps}}]\,(\mathsf{nl} \vee \mathsf{nm})$

PS2 $\mathsf{nm} \longrightarrow [+_{\mathsf{ps}}]\,(\mathsf{nm} \vee \mathsf{ns})$

PS3 $\mathsf{ns} \longrightarrow [+_{\mathsf{ps}}]\,(\mathsf{ns} \vee 0 \vee \mathsf{ps})$

PS4 $\mathsf{ps} \longrightarrow [+_{\mathsf{ps}}]\,(\mathsf{ps} \vee \mathsf{pm})$

PS5 $\mathsf{pm} \longrightarrow [+_{\mathsf{ps}}]\,(\mathsf{pm} \vee \mathsf{pl})$

PS6 $\mathsf{pl} \longrightarrow [+_{\mathsf{ps}}]\,\mathsf{pl}$

Z1 $\langle +_0 \rangle\,\varphi \longrightarrow [+_0]\,\varphi$

Z2 $[+_0]\,\varphi \longrightarrow \varphi$

Decidability and Completeness results

Theorem (Small Model Theorem)

Let φ a satisfiable formula, then φ is satisfied in a model with no more than $2^{|\varphi|}$ states.

Theorem

For every formula φ, if φ is a theorem then φ is valid.

Decidability and Completeness results

Theorem (Small Model Theorem)

Let φ a satisfiable formula, then φ is satisfied in a model with no more than $2^{|\varphi|}$ states.

Theorem

For every formula φ, if φ is a theorem then φ is valid.

Outline

Conclusions

- Different logics for order of magnitude reasoning have been introduced in order to deal with qualitative relations such as closeness and distance.
- Complete axiom systems for these logics have been defined.
- In the case of PDL, we have shown the decidability of the satisfiability problem.
- We have given relational proof system based on dual tableaux for our multimodal logic.

Future work

- We are trying to extend the PDL approach for more relations such as a linear order, by maintaining decidability and completeness.
- We have planned to give a relational proof system based on dual tableaux for our PDL logic.
- We are looking for more applications of our logics.

Thank you and Contact

THANK YOU VERY MUCH!!!

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Outline

The Language

Formulas:

- $\Phi_0 = \mathbb{V} \cup \mathbb{C}$, where $\mathbb{V}$ is a denumerable set of propositional variables and $\mathbb{C} = \{\mathsf{nl}, \mathsf{nm}, \mathsf{ns}, 0, \mathsf{ps}, \mathsf{pm}, \mathsf{pl}\}$.
- If φ and ψ are formulas and a is a program, then $\varphi \longrightarrow \psi$ (propositional implication), $\perp$ (propositional falsity) and $[a]\varphi$ (program neccesity) are also formulas.

Programs:

- $\Pi_0 = \{+_\star \mid \star \in \mathbb{C}\}$.
- If a and b are programs and φ is a formula, then $(a; b)$ ("do a followed by b"), $a \cup b$ ("do either a or b, nondeterministically"), a^* ("repeat a a nondeterministically chosen finite number of times") and $\varphi?$ ("proceed if φ is true, else fail") are also programs.

The Semantics

We now define the *semantics* of our logic. A *model* $\mathcal{M}$ is a tuple (W, m), where W is a non-empty set divided in 7 qualitative classes, $\{\mathrm{nl}, \mathrm{nm}, \mathrm{ns}, 0, \mathrm{ps}, \mathrm{pm}, \mathrm{pl}\}$ and m is a meaning function such that $m(p) \subseteq W$, for every propositional variable, $m(\star) = \star$, for every $\star \in \mathbb{C}$ and $m(a) \subseteq W \times W$, for all program a. Moreover, for every formula φ and ψ and for all programs a, b, we have:

- $m(\perp) = \varnothing \quad m(\varphi \longrightarrow \psi) = (W \smallsetminus m(\varphi)) \cup m(\psi)$
- $m([a]\varphi) = \{w \in W : \text{for all } v \in W, \text{ if } (w, v) \in m(a) \text{ then } v \in m(\varphi)\}$
- $m(a \cup b) = m(a) \cup m(b)$
- $m(a; b) = m(a); m(b)$
- $m(a^*) = m(a)^*$
- $m(\varphi?) = \{(w, w) : w \in m(\varphi)\}$

The properties of qualitative sum

1. $m(+_{\mathrm{ps}})(\mathrm{nl}) \subseteq \mathrm{nl} \cup \mathrm{nm}$
2. $m(+_{\mathrm{ps}})(\mathrm{nm}) \subseteq \mathrm{nm} \cup \mathrm{ns}$
3. $m(+_{\mathrm{ps}})(\mathrm{ns}) \subseteq \mathrm{ns} \cup 0 \cup \mathrm{ps}$
4. $m(+_{\mathrm{ps}})(\mathrm{ps}) \subseteq \mathrm{ps} \cup \mathrm{pm}$
5. $m(+_{\mathrm{ps}})(\mathrm{pm}) \subseteq \mathrm{pm} \cup \mathrm{pl}$
6. $m(+_{\mathrm{ps}})(\mathrm{pl}) \subseteq \mathrm{pl}$

The rest of axioms for qualitative sum

Positive medium numbers

PM1 $\mathsf{nl} \longrightarrow [+_{\mathsf{pm}}]\,(\mathsf{ns} \vee \mathsf{nm} \vee \mathsf{nl})$

PM2 $\mathsf{nm} \longrightarrow [+_{\mathsf{pm}}]\,(\mathsf{nm} \vee \mathsf{ns} \vee 0 \vee \mathsf{ps} \vee \mathsf{pm})$

PM3 $\mathsf{ns} \longrightarrow [+_{\mathsf{pm}}]\,(\mathsf{ps} \vee \mathsf{pm})$

PM4 $\mathsf{ps} \longrightarrow [+_{\mathsf{pm}}]\,(\mathsf{pm} \vee \mathsf{pl})$

PM5 $\mathsf{pm} \longrightarrow [+_{\mathsf{pm}}]\,(\mathsf{pm} \vee \mathsf{pl})$

PM6 $\mathsf{pl} \longrightarrow [+_{\mathsf{pm}}]\,\mathsf{pl}$

The rest of axioms for qualitative sum

Positive large numbers

PL1 $\mathsf{nm} \longrightarrow [+_{\mathsf{pl}}]\,(\mathsf{ps} \vee \mathsf{pm} \vee \mathsf{pl})$

PL2 $\mathsf{ns} \longrightarrow [+_{\mathsf{pl}}]\,(\mathsf{pm} \vee \mathsf{pl})$

PL3 $\mathsf{ps} \longrightarrow [+_{\mathsf{pl}}]\,\mathsf{pl}$

PL4 $\mathsf{pm} \longrightarrow [+_{\mathsf{pl}}]\,\mathsf{pl}$

PL5 $\mathsf{pl} \longrightarrow [+_{\mathsf{pl}}]\,\mathsf{pl}$

Fisher-Ladner Closure

$$FL : \Phi \longrightarrow 2^{\Phi} \quad FL^{\Box} : \{[a]\varphi \mid a \in \Pi, \varphi \in \Phi\} \longrightarrow 2^{\Phi}$$

(a) $FL(p) = \{p\}$, for every propositional variable p.

(b) $FL(\star) = \star$, for all $\star \in \mathbb{C}$.

(c) $FL(\varphi \longrightarrow \psi) = \{\varphi \longrightarrow \psi\} \cup FL(\varphi) \cup FL(\psi)$

(d) $FL(\bot) = \{\bot\}$

(e) $FL([a]\varphi) = FL^{\Box}([a]\varphi) \cup FL(\varphi)$

(f) $FL^{\Box}([a]\varphi) = \{[a]\varphi\}$, being a an atomic program.

(g) $FL^{\Box}([a \cup b]\varphi) = \{[a \cup b]\varphi\} \cup FL^{\Box}([a]\varphi) \cup FL^{\Box}([b]\varphi)$

(h) $FL^{\Box}([a;b]\varphi) = \{[a;b]\varphi\} \cup FL^{\Box}([a][b]\varphi) \cup FL^{\Box}([b]\varphi)$

(i) $FL^{\Box}([a^*]\varphi) = \{[a^*]\varphi\} \cup FL^{\Box}([a][a^*]\varphi)$

(j) $FL^{\Box}([\psi?]\varphi) = \{[\psi?]\varphi\} \cup FL(\psi)$

The filtration structure

The filtration structure $(\overline{W}, \overline{m})$ of (W, m) by $FL(\varphi)$ is defined on the quotient set $\overline{W} = W/\equiv$, being

$$u \equiv v \overset{\text{def}}{\Longleftrightarrow} \forall \psi \in FL(\varphi)[u \in m(\psi) \text{ iff } v \in m(\psi)]$$

Furthermore, the map $\overline{m}$ is defined as follows:

1. $\overline{m}(p) = \{\overline{u} \mid u \in m(p)\}$, for every propositional, variable p.
2. $\overline{m}(\star) = m(\star) = \star$, for all $\star \in \mathbb{C}$.
3. $\overline{m}(a) = \{(\overline{u}, \overline{v}) \mid \exists u' \in \overline{u} \text{ and } \exists v' \in \overline{v} \text{ such that } (u', v') \in m(a)\}$, for every atomic program a.

$\overline{m}$ is extended inductively to compound propositions and programs as described previously in the definition of model.

The Filtration Lemma

Lemma (Filtration Lemma)

Let (W, m) be a model and $(\overline{W}, \overline{m})$ defined as previously from a formula φ. Consider $u, v \in W$.

1. *For all $\psi \in FL(\varphi)$, $u \in m(\psi)$ iff $\overline{u} \in \overline{m}(\psi)$.*
2. *For all $[a]\psi \in FL(\varphi)$,*
   - *if $(u, v) \in m(a)$ then $(\overline{u}, \overline{v}) \in \overline{m}(a)$;*
   - *if $(\overline{u}, \overline{v}) \in \overline{m}(a)$ and $u \in m([a]\psi)$, then $v \in m(\psi)$.*

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