Tomography Using Accretive Operators

J. Garcia Falset

U. València

Sevilla, Abril de 2010

1. Introducción
   - X-ray measurement models
   - The Evolution Equation

2. Existence of solution
   - Preliminaires
   - General Theory
   - Existence of Solution for the method
   - Asymptotic behavior

In medical X-ray tomography, the inner structure of a patient is reconstructed from a collection of projection images. the widely used computerized tomography (CT) imaging uses an extensive set of projections acquired from all around the body. This type of reconstruction is well understood, the most popular method being filtered back-projection (FBP).

In medical X-ray tomography, the inner structure of a patient is reconstructed from a collection of projection images. the widely used computerized tomography (CT) imaging uses an extensive set of projections acquired from all around the body. This type of reconstruction is well understood, the most popular method being filtered back-projection (FBP).

Nevertheless, there are many clinical applications where three-dimensional information is helpful, but a complete projection data set is not available. For instance, in mammography and intraoral dental imaging, the X-ray detector is in a fixed position behind the tissue, and the X-ray source moves with respect to the detector. In this cases the projections can be taken from a view angle significantly less than 180^o, leading to a limited angle tomography problem. In some applications, such as the radiation dose to the patient is minimized by keeping the number of projection small. In addition, the projections are typically truncated to detector size, yielding a local tomography problem. We refer to the above types of incomplete data as sparse projection data, (SPD).

Sparse projection data does not contain information enough in order to describe completely the tissue, and therefore successful reconstruction requires some form of regularization or priori information.

Sparse projection data does not contain information enough in order to describe completely the tissue, and therefore successful reconstruction requires some form of regularization or priori information.

For instance, in

Natterer, F. *The mathematics of computerized tomography, John Wiley and Sons, Chichester, UK, Teubner, Stuttgart, 1986.*

is proved that methods such as (FBP) are not suited for sparse projection data.

Sparse projection data does not contain information enough in order to describe completely the tissue, and therefore successful reconstruction requires some form of regularization or priori information.

For instance, in

Natterer, F. *The mathematics of computerized tomography, John Wiley and Sons, Chichester, UK, Teubner, Stuttgart, 1986.*

is proved that methods such as (FBP) are not suited for sparse projection data.

More promising approaches include, among others, total variation methods, variational methods and deformable models.

See with respect to total variational methods

Candès, E.J.; Romberg, J.; Tao, T. *Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inform. Theory,* **52** *(2006), 498-509.*

See with respect to total variational methods

Candès, E.J.; Romberg, J.; Tao, T. *Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inform. Theory,* **52** *(2006), 498-509.*

variational methods

Kybic, J.; Blu, T.; Unser, M. *Variational approach to tomographic reconstruction, Proc. of SPIE (San Diego, 2001),* **4322** *Medical Imaging 2001: Image Proceeding, Milan Sonka, Kenneth M. Hanson, eds., (2001) 30-39.*

See with respect to total variational methods

Candès, E.J.; Romberg, J.; Tao, T. *Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inform. Theory,* **52** *(2006), 498-509.*

variational methods

Kybic, J.; Blu, T.; Unser, M. *Variational approach to tomographic reconstruction, Proc. of SPIE (San Diego, 2001),* **4322** *Medical Imaging 2001: Image Proceeding, Milan Sonka, Kenneth M. Hanson, eds., (2001) 30-39.*

deformable models

Yu, D.F.; Fessler, J.A. *Edge-preserving tomographic reconstruction with nonlocal regularization,* IEEE Trans. Medi. Imaging **21**, (2002) 159-173.

We study a variant of the level set method, where the X-ray attenuation coefficient is modeled as the function $\max\{\Phi(x), 0\}$ with Φ a smooth function. Thus we make use of the natural a priori information that the X-ray attenuation coefficient is always non negative (The intensity of X-ray does not increase inside tissue).

We study a variant of the level set method, where the X-ray attenuation coefficient is modeled as the function $\max\{\Phi(x), 0\}$ with Φ a smooth function. Thus we make use of the natural a priori information that the X-ray attenuation coefficient is always non negative (The intensity of X-ray does not increase inside tissue).

This method has been developed by

Kolehmainen, V.; Lassas, M.; Siltanen, S. *Limited data X-ray tomographi using nonlinear evolution equations.* SIAM J. Sci. Comput. **30**(3) (2008) 1413-1429.

We assume that the attenuation coefficient $v \in L^2(\Omega)$ for a bounded subset $\Omega \subseteq \mathbb{R}^2$ and use the following model for the direct problem:

We assume that the attenuation coefficient $v \in L^2(\Omega)$ for a bounded subset $\Omega \subseteq \mathbb{R}^2$ and use the following model for the direct problem:

$$m = A(v) + \varepsilon, \tag{1}$$

We assume that the attenuation coefficient $v \in L^2(\Omega)$ for a bounded subset $\Omega \subseteq \mathbb{R}^2$ and use the following model for the direct problem:

$$m = A(v) + \varepsilon, \tag{1}$$

where A is a linear operator on $L^2(\Omega)$ with appropriate target space and ε is a measurement of the noise.

We assume that the attenuation coefficient $v \in L^2(\Omega)$ for a bounded subset $\Omega \subseteq \mathbb{R}^2$ and use the following model for the direct problem:

$$m = A(v) + \varepsilon, \tag{1}$$

where A is a linear operator on $L^2(\Omega)$ with appropriate target space and ε is a measurement of the noise.

The idea is to reconstruct v approximately from m.

In medical X-ray imaging, an X-ray source is placed on one side of the target tissue. Radiation passes through the tissue, and the attenuated signal is detected on the other side.

In medical X-ray imaging, an X-ray source is placed on one side of the target tissue. Radiation passes through the tissue, and the attenuated signal is detected on the other side.

A two-dimensional model slice through the target tissue by a rectangle $\Omega \subseteq \mathbb{R}^2$ and a nonnegative attenuation coefficient $v : \Omega \to [0, \infty)$. The tissue is contained in a subset $\Omega_1 \subset \Omega$, and $v(x) = 0$ for $x \in \Omega \setminus \Omega_1$.

In medical X-ray imaging, an X-ray source is placed on one side of the target tissue. Radiation passes through the tissue, and the attenuated signal is detected on the other side.

A two-dimensional model slice through the target tissue by a rectangle $\Omega \subseteq \mathbb{R}^2$ and a nonnegative attenuation coefficient $v : \Omega \to [0, \infty)$. The tissue is contained in a subset $\Omega_1 \subset \Omega$, and $v(x) = 0$ for $x \in \Omega \setminus \Omega_1$. This yields the linear model

$$\int_L v(x)dx = \lg(I_0) - \lg(I_1), \qquad (2)$$

where L is the line of the X-ray, I_0 is the initial intensity of the X-ray beam when entering Ω and I_1 is the attenuated intensity at the detector.

Radon transform

Define the operator $A : L^2(\Omega) \to L^2(D)$ appearing in $m = A(v) + \varepsilon$ by

Radon transform

Define the operator $A : L^2(\Omega) \to L^2(D)$ appearing in $m = A(v) + \varepsilon$ by

$$(Av)(\theta, s) = \int_{L(\theta,s)} v(x)dx,$$

where $L(\theta, s) := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \cos(\theta) + x_2 \sin(\theta) = s\}$.

Radon transform

Define the operator $A : L^2(\Omega) \to L^2(D)$ appearing in $m = A(v) + \varepsilon$ by

$$(Av)(\theta, s) = \int_{L(\theta,s)} v(x)dx,$$

where $L(\theta, s) := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \cos(\theta) + x_2 \sin(\theta) = s\}$.
It is allowed models of limited angle and local tomography by taking

$$D := \{(\theta, s) : \ \theta \in [\theta_0, \theta_1], \quad s \in [s_0(\theta), s_1(\theta)]\},$$

where $0 \leq \theta_0 < \theta_1 \leq 2\pi$ and $-\infty < s_0(\theta) < s_1(\theta) < +\infty$. Finally, we assume that $\varepsilon \in L^2(D)$.

Pencil beam model

Suppose we have N_1 projections images with a digital detector consisting of N_2 pixels.

Pencil beam model

Suppose we have N_1 projections images with a digital detector consisting of N_2 pixels.

Then our data consists of integral of v over $N = N_1 N_2$ different lines L in $\int_L v(x)dx = \lg(I_0) - \lg(I_1)$. According, the operator in $m = A(v) + \varepsilon$ is defined as

$$A : L^2(\Omega) \to \mathbb{R}^N,$$

the measurement is a vector $m \in \mathbb{R}^N$, and noise is modeled by a Gaussian zero-centered random vector ε taking values in $\mathbb{R}^N$.

Consider the measurement given by equation

$$m = A(v) + \varepsilon,$$

Consider the measurement given by equation

$$m = A(v) + \varepsilon,$$

In the case that the X-ray attenuation coefficient v is smooth and non null only inside a subset $\Omega_1 \subset \Omega$.

Consider the measurement given by equation

$$m = A(v) + \varepsilon,$$

In the case that the X-ray attenuation coefficient v is smooth and non null only inside a subset $\Omega_1 \subset \Omega$.

we design an algorithm that

(i) Constructs an approximation Ω_2 for the subset Ω_1, and

(ii) with given approximation Ω_2 produces a reconstruction w that solves the Tikhonov regularization problem

$$w = argmin_u \left\{ \frac{1}{2} \|A(u) - m\|^2_{L^2(D)} + \frac{\beta}{2} \int_\Omega \langle \nabla u, \nabla u \rangle dx \right\},$$

Consider the measurement given by equation

$$m = A(v) + \varepsilon,$$

In the case that the X-ray attenuation coefficient v is smooth and non null only inside a subset $\Omega_1 \subset \Omega$.

we design an algorithm that

(i) Constructs an approximation Ω_2 for the subset Ω_1, and

(ii) with given approximation Ω_2 produces a reconstruction w that solves the Tikhonov regularization problem

$$w = argmin_u \left\{ \frac{1}{2} \|A(u) - m\|^2_{L^2(D)} + \frac{\beta}{2} \int_\Omega \langle \nabla u, \nabla u \rangle dx \right\},$$

where $\beta > 0$ is a parameter and the minimum is taken over all u satisfying

(a) $u|_{\Omega \setminus \Omega_2} \equiv 0,$

(b) $u|_{\Omega_2} \in H^1_0(\Omega) = \{g \in L^2(\Omega) : \frac{\partial g}{\partial x_i} \in L^2(\Omega),\ i = 1, 2;\ g|_{\partial \Omega_2} = 0\}.$

Formulation of this method

Tikhonov regularization yields the cost functional

$$F(u) = \frac{1}{2}\|A(f(u)) - m\|^2_{L^2(D)} + \frac{\beta}{2}\int_\Omega \langle \nabla u, \nabla u\rangle dx. \tag{3}$$

Formulation of this method

Tikhonov regularization yields the cost functional

$$F(u) = \frac{1}{2}\|A(f(u)) - m\|^2_{L^2(D)} + \frac{\beta}{2}\int_\Omega \langle \nabla u, \nabla u\rangle dx. \tag{3}$$

The cutoff function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(s) = \begin{cases} s, & s > 0 \\ 0, & s \leq 0 \end{cases} \tag{4}$$

Tikhonov regularization yields the cost functional

$$F(u) = \frac{1}{2}\|A(f(u)) - m\|^2_{L^2(D)} + \frac{\beta}{2}\int_\Omega \langle \nabla u, \nabla u\rangle dx. \tag{5}$$

If now we consider the solution, if it exists, of the evolution equation:

$$\begin{array}{l} \partial_t \phi(x,t) = -H(\phi)A^*(A(f(\phi(x,t))) - m) + \beta \triangle \phi(x,t), \\ (\partial_\nu \phi(x,t) - r\phi(x,t))\,|_{\partial\Omega} = 0, \\ \phi(x,0) = \phi_0(x), \end{array} \tag{6}$$

with ν the interior normal of $\partial\Omega$, $\beta > 0$ a regularization parameter, and $r \geq 0$.

Tikhonov regularization yields the cost functional

$$F(u) = \frac{1}{2}\|A(f(u)) - m\|^2_{L^2(D)} + \frac{\beta}{2}\int_\Omega \langle \nabla u, \nabla u\rangle dx. \qquad (5)$$

If now we consider the solution, if it exists, of the evolution equation:

$$\begin{aligned} &\partial_t\phi(x,t) = -H(\phi)A^*(A(f(\phi(x,t))) - m) + \beta\triangle\phi(x,t), \\ &(\partial_\nu\phi(x,t) - r\phi(x,t))\,|_{\partial\Omega} = 0, \\ &\phi(x,0) = \phi_0(x), \end{aligned} \qquad (6)$$

with ν the interior normal of $\partial\Omega$, $\beta > 0$ a regularization parameter, and $r \geq 0$.

Computing the derivative $\partial_t F(\phi)$, we obtain

$$\partial_t F(\phi) = -\int_\Omega (A^*(A(f(\phi)) - m) - \beta\triangle\phi)^2 dx \leq 0.$$

evolution equation

$$\begin{aligned} &\partial_t \phi(x,t) = -A^*(A(f(\phi(x,t))) - m) + \beta \triangle \phi(x,t), \\ &(\partial_\nu \phi(x,t) - r\phi(x,t))\,|_{\partial\Omega} = 0, \\ &\phi(x,0) = \phi_0(x), \end{aligned} \tag{7}$$

with ν the interior normal of $\partial\Omega$, $\beta > 0$ a regularization parameter, and $r \geq 0$.

evolution equation

$$\begin{aligned} &\partial_t \phi(x,t) = -A^*(A(f(\phi(x,t))) - m) + \beta \triangle \phi(x,t), \\ &(\partial_\nu \phi(x,t) - r\phi(x,t))\,|_{\partial\Omega} = 0, \\ &\phi(x,0) = \phi_0(x), \end{aligned} \tag{7}$$

with ν the interior normal of $\partial\Omega$, $\beta > 0$ a regularization parameter, and $r \geq 0$.

Therefore, we have to study if there exists a function Φ such that the function w satisfies

$$F(\Phi) = \lim_{t\to\infty} F(\phi(x,t)) \tag{8}$$

evolution equation

$$
\begin{array}{l}
\partial_t \phi(x,t) = -A^*(A(f(\phi(x,t))) - m) + \beta \triangle \phi(x,t), \\
(\partial_\nu \phi(x,t) - r\phi(x,t))\,|_{\partial\Omega} = 0, \\
\phi(x,0) = \phi_0(x),
\end{array} \quad (7)
$$

with ν the interior normal of $\partial\Omega$, $\beta > 0$ a regularization parameter, and $r \geq 0$.

Therefore, we have to study if there exists a function Φ such that the function w satisfies

$$
F(\Phi) = \lim_{t\to\infty} F(\phi(x,t)) \quad (8)
$$

in this case, we approximate the X-ray attenuation coefficient v by

$$
w = f(\Phi)
$$

.

An operator A on X is said to be *accretive* if the inequality $\|x - y + \lambda(z - w)\| \geq \|x - y\|$ holds for all $\lambda \geq 0$, $(x, z); (y, w) \in A$.

An operator A on X is said to be *accretive* if the inequality $\|x - y + \lambda(z - w)\| \geq \|x - y\|$ holds for all $\lambda \geq 0$, $(x, z); (y, w) \in A$. If, in addition, $\mathcal{R}(I + \lambda A)$ is for one, hence for all, $\lambda > 0$, precisely X, then A is called m-accretive.

An operator A on X is said to be *accretive* if the inequality $\|x-y+\lambda(z-w)\| \geq \|x-y\|$ holds for all $\lambda \geq 0$, $(x,z);(y,w) \in A$. If, in addition, $\mathcal{R}(I+\lambda A)$ is for one, hence for all, $\lambda > 0$, precisely X, then A is called m-accretive.

- Browder, Felix E. *Nonlinear mappings of nonexpansive and accretive type in Banach spaces.* Bull. Amer. Math. Soc. **7**3 1967, 875–882.
- Kato, T. *Nonlinear semigroups and evolution equations.* J. Math. Soc. Japan **19** (1967) 508–520.

An operator A on X is said to be *accretive* if the inequality $\|x-y+\lambda(z-w)\| \geq \|x-y\|$ holds for all $\lambda \geq 0$, $(x,z);(y,w) \in A$. If, in addition, $\mathcal{R}(I+\lambda A)$ is for one, hence for all, $\lambda > 0$, precisely X, then A is called m-accretive.

- Browder, Felix E. *Nonlinear mappings of nonexpansive and accretive type in Banach spaces.* Bull. Amer. Math. Soc. **7**3 1967, 875–882.
- Kato, T. *Nonlinear semigroups and evolution equations.* J. Math. Soc. Japan **19** (1967) 508–520.

Those accretive operators which are m-accretive play an important role in the study of nonlinear partial differential equations.

Consider the Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t), \ t \in (0, T), \\ u(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{9}$$

where A is m-accretive on X and $f \in L^1(0, T, X)$.

Consider the Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t), \ t \in (0, T), \\ u(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{9}$$

where A is m-accretive on X and $f \in L^1(0, T, X)$.

It is well known that (10) has a unique integral solution in the sense of Bénilan

Ph. Bénilan, *Équations d'évolution dans un espace de Banach quelconque et applications,* Thèse de doctorat d'État, Orsay, 1972.

Consider the Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t), \ t \in (0, T), \\ u(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{10}$$

where A is m-accretive on X and $f \in L^1(0, T, X)$.

It is well known that (10) has a unique integral solution in the sense of Bénilan

There exists a unique continuous function $u : [0, T] \to \overline{D(A)}$ such that $u(0) = x_0$, and moreover, for each $(x, y) \in A$ and $0 \leq s \leq t \leq T$, we have

$$\|u(t) - x\|^2 - \|u(s) - x\|^2 \leq 2 \int_s^t \langle f(\tau) - y, u(\tau) - x \rangle_+ d\tau. \tag{11}$$

Here the function $\langle \cdot, \cdot \rangle_+ : X \times X \to \mathbb{R}$ is defined by $\langle y, x \rangle_+ = \sup\{x^*(y) \ : \ x^* \in J(x)\}$, where $J : X \to 2^{X^*}$ is the duality mapping on X, *i.e.*, $J(x) = \{x^* \in X^* \ : \ x^*(x) = \|x\|^2, \|x^*\| = \|x\|\}$.

A *strong solution* of Problem (10) is a function $u \in W^{1,\infty}(0, T; X)$, *i.e.*, u is locally absolutely continuous and almost differentiable everywhere, $u' \in L^{\infty}(0, T; X)$, and $u'(t) + A(u(t)) \ni f(t)$ for almost all $t \in [0, T]$.

A *strong solution* of Problem (10) is a function $u \in W^{1,\infty}(0, T; X)$, *i.e.*, u is locally absolutely continuous and almost differentiable everywhere, $u' \in L^\infty(0, T; X)$, and $u'(t) + A(u(t)) \ni f(t)$ for almost all $t \in [0, T]$.

Theorem

If X is a Banach space with the Radon-Nikodym property, $A : D(A) \subseteq X \to 2^X$ is an m-accretive operator, and $f \in BV(0, T; X)$, i.e., f is a function of bounded variation on $[0, T]$, then Problem (10) has a unique strong solution whenever $x_0 \in D(A)$.

we say that $u \in C(0, T; X)$ is a *weak solution* of Problem (10) if there are sequences $(u_n) \subseteq W^{1,\infty}(0, T; X)$ and $(f_n) \subseteq L^1(0, T; X)$ satisfying the following four conditions:

we say that $u \in C(0, T; X)$ is a *weak solution* of Problem (10) if there are sequences $(u_n) \subseteq W^{1,\infty}(0, T; X)$ and $(f_n) \subseteq L^1(0, T; X)$ satisfying the following four conditions:

1. $u_n'(t) + Au_n(t) \ni f_n(t)$ for almost all $t \in [0, T]$, $n = 1, 2, ...;$

we say that $u \in C(0,T;X)$ is a *weak solution* of Problem (10) if there are sequences $(u_n) \subseteq W^{1,\infty}(0,T;X)$ and $(f_n) \subseteq L^1(0,T;X)$ satisfying the following four conditions:

1. $u_n'(t) + Au_n(t) \ni f_n(t)$ for almost all $t \in [0,T]$, $n = 1, 2, \ldots;$
2. $\lim_{n\to\infty} \|u_n - u\|_\infty = 0;$

we say that $u \in C(0,T;X)$ is a *weak solution* of Problem (10) if there are sequences $(u_n) \subseteq W^{1,\infty}(0,T;X)$ and $(f_n) \subseteq L^1(0,T;X)$ satisfying the following four conditions:

1. $u_n'(t) + Au_n(t) \ni f_n(t)$ for almost all $t \in [0,T]$, $n = 1,2,\ldots;$
2. $\lim_{n\to\infty} \|u_n - u\|_\infty = 0;$
3. $u(0) = x_0;$

we say that $u \in C(0, T; X)$ is a *weak solution* of Problem (10) if there are sequences $(u_n) \subseteq W^{1,\infty}(0, T; X)$ and $(f_n) \subseteq L^1(0, T; X)$ satisfying the following four conditions:

1. $u_n'(t) + Au_n(t) \ni f_n(t)$ for almost all $t \in [0, T]$, $n = 1, 2, ...;$
2. $\lim_{n\to\infty} \|u_n - u\|_\infty = 0;$
3. $u(0) = x_0;$
4. $\lim_{n\to\infty} \|f_n - f\|_1 = 0;$

we say that $u \in C(0, T; X)$ is a *weak solution* of Problem (10) if there are sequences $(u_n) \subseteq W^{1,\infty}(0, T; X)$ and $(f_n) \subseteq L^1(0, T; X)$ satisfying the following four conditions:

1. $u_n'(t) + Au_n(t) \ni f_n(t)$ for almost all $t \in [0, T]$, $n = 1, 2, \ldots$;
2. $\lim_{n\to\infty} \|u_n - u\|_\infty = 0$;
3. $u(0) = x_0$;
4. $\lim_{n\to\infty} \|f_n - f\|_1 = 0$;

Theorem

Let X be a Banach space with the Radon-Nikodym property. Then Problem (10) admits a unique weak solution which is the unique integral solution of this problem.

Theorem

Let E be a real Banach space. Consider $A : D(A) \subseteq E \to 2^E$ an m-accretive operator on E. Let $B : E \to E$ be a k-Lipschitzian mapping. Then the Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni B(u(t)), \ t \in (0, +\infty), \\ u(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{12}$$

has a unique integral solution.

Theorem

Let E be a real Banach space. Consider $A : D(A) \subseteq E \to 2^E$ an m-accretive operator on E. Let $B : E \to E$ be a k-Lipschitzian mapping. Then the Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni B(u(t)), \ t \in (0, +\infty), \\ u(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{12}$$

has a unique integral solution.

Remark

This type of results has been studied for example in
Garcia-Falset, Jesús; Reich, Simeon, *Integral solutions to a class of nonlocal evolution equations*, CCM (to appear).

Theorem

Let E be a real Banach space. Consider $A : D(A) \subseteq E \to 2^E$ an m-accretive operator on E. Let $B : E \to E$ be a k-Lipschitzian mapping. Then the Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni B(u(t)), \; t \in (0, +\infty), \\ u(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{13}$$

has a unique integral solution.

Theorem

Let E be a real Banach space. Consider $A : D(A) \subseteq E \to 2^E$ an m-accretive operator on E. Let $B : E \to E$ be a k-Lipschitzian mapping. Then the Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni B(u(t)), \ t \in (0, +\infty), \\ u(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{13}$$

has a unique integral solution.

Theorem

Let E be a Banach space with Radon-Nikodym property (RN for short). Under the assumptions of Theorem 5, if we define the Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni B(u(t)) \\ u(0) = u_0 \in \overline{D(\mathcal{A})} \end{cases} \tag{14}$$

Then, it has a unique weak solution.

Definition

Let $\phi : [0, \infty[\to [0, \infty[$ be a continuous function such that $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$. Let X be a Banach space. An operator $A : D(A) \to 2^X$ is said to be ϕ-strongly accretive if for every $(x, u), (y, v) \in A$, then

$$\phi(\|x - y\|)\|x - y\| \leq \langle u - v, x - y \rangle_+.$$

Definition

Let $\phi : [0,\infty[\to [0,\infty[$ be a continuous function such that $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$. Let X be a Banach space. An operator $A : D(A) \to 2^X$ is said to be ϕ-strongly accretive if for every $(x,u),(y,v) \in A$, then

$$\phi(\|x-y\|)\|x-y\| \leq \langle u-v, x-y\rangle_+.$$

Definition

Let $\phi : [0,\infty[\to [0,\infty[$ be a either continuous or nondecresing function such that $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$. A mapping $A : D(A) \to 2^X$ is said to be ϕ-expansive if for every $x, y \in D(A)$ and every $u \in A(x)$, and $v \in A(y)$, then

$$\|u-v\| \geq \phi(\|x-y\|). \tag{15}$$

Remark

The main result of
Garcia-Falset, J.; Morales, Cl. *Existence theorems for m-accretive operators in Banach spaces.* J. Math. Anal. Appl. **309** (2005), 453–461.
establish that if X is a Banach space and $A : D(A) \to 2^X$ is an m-accretive and ϕ-expansive operator, then A is surjective.

Definition

Let E be a Banach space, let $\phi : E \rightarrow [0, \infty)$ be a continuous function such that $\phi(0) = 0$, $\phi(x) > 0$ for $x \neq 0$ and which satifies the following condition:
For every sequence (x_n) in E such that $(\|x_n\|)$ is decreasing and $\phi(x_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\|x_n\| \rightarrow 0$.
An accretive operator $A : D(A) \rightarrow 2^E$ with $0 \in Az$ is said to be ϕ-accretive at zero whenever the inequality

$$\langle u, x - z \rangle_+ \geq \phi(x - z), \text{ for all } (x, u) \in A, \tag{16}$$

holds.

Definition

Let E be a Banach space, let $\phi : E \to [0, \infty)$ be a continuous function such that $\phi(0) = 0$, $\phi(x) > 0$ for $x \neq 0$ and which satifies the following condition:
For every sequence (x_n) in E such that $(\|x_n\|)$ is decreasing and $\phi(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$.
An accretive operator $A : D(A) \to 2^E$ with $0 \in Az$ is said to be ϕ-accretive at zero whenever the inequality

$$\langle u, x - z\rangle_+ \geq \phi(x - z), \text{ for all } (x, u) \in A, \tag{16}$$

holds.

Remark

The uniqueness of a zero for an operator either ϕ-expansive or ϕ-accretive at zero is an immediate consequence of (15) or (16), respectively.

Proposition 3.4 and Remark 4.5 of
Garcia-Falset, J. *Strong convergence theorems for resolvents of accretive operators*. Fixed Point theory and its applications, Yokohama Publishers. (2005), 87–94.
prove that every m-ψ-strongly accretive operator is both ψ-expansive and ϕ-accretive at zero with $\phi = \psi \circ \| \cdot \|$.

Proposition 3.4 and Remark 4.5 of
Garcia-Falset, J. *Strong convergence theorems for resolvents of accretive operators*. Fixed Point theory and its applications, Yokohama Publishers. (2005), 87–94.
prove that every *m*-ψ-strongly accretive operator is both ψ-expansive and ϕ-accretive at zero with $\phi = \psi \circ \|\cdot\|$.
Finally, in the above paper is also proved that there is not any relationship between to be ϕ-expansive and to be ϕ-accretive at zero.

Theorem

Let E be a Banach space with RN property. Consider $P : D(P) \subseteq E \to E$ an m-ψ-strongly accretive operator on E. Assume that u_0 is an element of $\overline{D(P)}$, and $h \in E$. If $u : [0, \infty) \to \overline{D(P)}$ is the unique weak solution of the Cauchy problem

$$\begin{cases} u'(t) + \mathcal{H}(u(t)) = 0 \\ u(0) = u_0 \in \overline{D(P)}, \end{cases} \tag{17}$$

where $\mathcal{H} = P - h$. Then $\lim_{t\to+\infty} u(t) = z$, being z the unique element in $D(P)$ such that $h = P(z)$.

Theorem

Let E be a Banach space with RN property. Consider $P : D(P) \subseteq E \to E$ an m-ψ-strongly accretive operator on E. Assume that u_0 is an element of $\overline{D(P)}$, and $h \in E$. If $u : [0, \infty) \to \overline{D(P)}$ is the unique weak solution of the Cauchy problem

$$\begin{cases} u'(t) + \mathcal{H}(u(t)) = 0 \\ u(0) = u_0 \in \overline{D(P)}, \end{cases} \tag{17}$$

where $\mathcal{H} = P - h$. Then $\lim_{t\to+\infty} u(t) = z$, being z the unique element in $D(P)$ such that $h = P(z)$.

The above result is a particular case of Corollary 9 of Garcia-Falset, J. *The asymptotic behavior of the solutions of the Cauchy problem generated by ϕ-accretive operators.* J. Math. Anal. Appl. **310** (2005) 594-608.

If we denote $g = A^*(m)$. Consider the initial boundary value Problem

$$\begin{cases} \partial_t \phi(x,t) = -A^*(A(f(\phi(x,t))) + \beta \triangle \phi(x,t) + g, \\ (\partial_\nu \phi(x,t) - r\phi(x,t))\,|_{\partial\Omega} = 0, \\ \phi(x,0) = \phi_0(x), \end{cases} \tag{18}$$

If we denote $g = A^*(m)$. Consider the initial boundary value Problem

$$\begin{cases} \partial_t \phi(x,t) = -A^*(A(f(\phi(x,t))) + \beta \triangle \phi(x,t) + g, \\ (\partial_\nu \phi(x,t) - r\phi(x,t))\,|_{\partial\Omega} = 0, \\ \phi(x,0) = \phi_0(x), \end{cases} \tag{18}$$

It is well known that if $\beta > 0$, and we consider the function $j : \mathbb{R} \to \mathbb{R}$ given by $j(s) = \frac{r}{2}s^2$ and define the function $\varphi : L^2(\Omega) \to]-\infty, +\infty]$ by

$$\varphi(u) = \begin{cases} \frac{\beta}{2}\int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} j(u)dx, \;\; u \in W^{1,2}(\Omega), \;\; j(u) \in L^1(\partial\Omega), \\ +\infty, \;\; \text{otherwise} \end{cases} \tag{19}$$

Then, φ is a proper lower semi continuous convex function in $L^2(\Omega)$ such that $D(\varphi) = W^{1,2}(\Omega)$ and moreover, its subdifferential is given by

If we denote $g = A^*(m)$. Consider the initial boundary value Problem

$$\left\{ \begin{array}{l} \partial_t \phi(x,t) = -A^*(A(f(\phi(x,t))) + \beta \triangle \phi(x,t) + g, \\ (\partial_\nu \phi(x,t) - r\phi(x,t))\,|_{\partial\Omega} = 0, \\ \phi(x,0) = \phi_0(x), \end{array} \right. \tag{18}$$

It is well known that if $\beta > 0$, and we consider the function $j : \mathbb{R} \to \mathbb{R}$ given by $j(s) = \frac{r}{2}s^2$ and define the function $\varphi : L^2(\Omega) \to] - \infty, +\infty]$ by

$$\varphi(u) = \left\{ \begin{array}{l} \frac{\beta}{2} \int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} j(u) dx, \;\; u \in W^{1,2}(\Omega), \;\; j(u) \in L^1(\partial\Omega), \\ +\infty, \;\; \text{otherwise} \end{array} \right. \tag{19}$$

Then, φ is a proper lower semi continuous convex function in $L^2(\Omega)$ such that $D(\varphi) = W^{1,2}(\Omega)$ and moreover, its subdifferential is given by

$$\partial\varphi(u) = -\beta\triangle(u), \text{for all } u \in D(\partial\varphi)$$

where $D(\partial\varphi) = \{u \in W^{2,2}(\Omega) : \;\; \frac{\partial}{\partial\nu} u - ru = 0, \;\text{a.e. on } \partial\Omega\}.$

It is well known that if $\beta > 0$, and we consider the function $j : \mathbb{R} \to \mathbb{R}$ given by $j(s) = \frac{r}{2}s^2$ and define the function $\varphi : L^2(\Omega) \to]-\infty, +\infty]$ by

$$\varphi(u) = \begin{cases} \frac{\beta}{2}\int_\Omega |\nabla u|^2 dx + \beta\int_{\partial\Omega} j(u)dx, & u \in W^{1,2}(\Omega),\ j(u) \in L^1(\partial\Omega), \\ +\infty, & \text{otherwise} \end{cases} \tag{20}$$

Then, φ is a proper lower semi continuous convex function in $L^2(\Omega)$ such that $D(\varphi) = W^{1,2}(\Omega)$ and moreover, its subdifferential is given by

$$\partial\varphi(u) = -\beta\triangle(u), \text{for all } u \in D(\partial\varphi)$$

where $D(\partial\varphi) = \{u \in W^{2,2}(\Omega) : \ \frac{\partial}{\partial\nu}u - ru = 0, \text{ a.e. on } \partial\Omega\}$.
V. Barbu, *Nonlinear differential equations of monotone types in Banach spaces,* Springer (2010).

rewrite the equation as a Cauchy Problem

Consider $u(t) := \psi(., t) \in \{u \in W^{2,2}(\Omega) : \ \partial_\nu u = ru \ a.\ e.\ on\ \partial\Omega\}$,
$g = A^*(m) \in L^2(\Omega)$,

$$\begin{cases} \mathcal{A}(u(t)) = -\beta\Delta(u(t)), \\ B(u(t)) = -A^*Af(u(t)) + g, \end{cases} \tag{21}$$

rewrite the equation as a Cauchy Problem

Consider $u(t) := \psi(., t) \in \{u \in W^{2,2}(\Omega) : \ \partial_\nu u = ru \ a.\ e.\ on\ \partial\Omega\}$,
$g = A^*(m) \in L^2(\Omega)$,

$$\begin{cases} \mathcal{A}(u(t)) = -\beta\Delta(u(t)), \\ B(u(t)) = -A^*Af(u(t)) + g, \end{cases} \tag{21}$$

We interpret and rewrite Problem (18) as follow:

$$\begin{cases} u'(t) + \mathcal{A}(u(t)) = B(u(t)), \ \ t > 0 \\ u(0) = \phi_0. \end{cases} \tag{22}$$

Strong solution

Theorem

Let Ω be a bounded subset of $\mathbb{R}^2$ with smooth boundary. Let $A : L^2(\Omega) \to L^2(D)$ be a continuous linear operator, where D is either a subset of $\mathbb{R}^2$ equipped with the Lebesgue measure, or $D = \{1, 2, ..., N\}$ equipped with the counting measure. If we define $B : L^2(\Omega) \to L^2(\Omega)$ by $B(u) = -A^*(A(f(u))) + g$, Then Problem

$$\begin{cases} u'(t) + \mathcal{A}(u(t)) = B(u(t)), \;\; t > 0 \\ u(0) = \phi_0. \end{cases} \tag{23}$$

has a unique strong solution whenever $\phi_0 \in L^2(\Omega)$.

Proof

- It is clear that $B : L^2(\Omega) \to L^2(\Omega)$ is a k-lipschitzian.

Proof

- It is clear that $B : L^2(\Omega) \to L^2(\Omega)$ is a k-lipschitzian.
- by the above comments the operator $-\beta\Delta$ is m-accretive on $L^2(\Omega)$ when its domain is given by $\{u \in W^{2,2}(\Omega) : \ \partial_\nu u = ru \ a.\ e.\ on\ \partial\Omega\}$.

Proof

- It is clear that $B : L^2(\Omega) \to L^2(\Omega)$ is a k-lipschitzian.
- by the above comments the operator $-\beta\Delta$ is m-accretive on $L^2(\Omega)$ when its domain is given by $\{u \in W^{2,2}(\Omega) : \ \partial_\nu u = ru \ a.\ e.\ on\ \partial\Omega\}$.

Under these conditions Theorems 6 allow us to conclude that Problem (22) has a unique weak solution.

Proof

- It is clear that $B : L^2(\Omega) \to L^2(\Omega)$ is a k-lipschitzian.
- by the above comments the operator $-\beta\Delta$ is m-accretive on $L^2(\Omega)$ when its domain is given by $\{u \in W^{2,2}(\Omega) : \ \partial_\nu u = ru \ a.\ e.\ on\ \partial\Omega\}$.

Under these conditions Theorems 6 allow us to conclude that Problem (22) has a unique weak solution.

let w be the solution of the problem, in this case we can consider the function $B(w(\cdot))$, since B is k-lipschitzian and $B(0) = 0$, it is clear that $B(w(\cdot)) \in L^2(0, T; L^2(\Omega))$ for all $T > 0$.

Proof

- It is clear that $B : L^2(\Omega) \to L^2(\Omega)$ is a k-lipschitzian.
- by the above comments the operator $-\beta\Delta$ is m-accretive on $L^2(\Omega)$ when its domain is given by $\{u \in W^{2,2}(\Omega) : \partial_\nu u = ru \text{ a. e. on } \partial\Omega\}$.

Under these conditions Theorems 6 allow us to conclude that Problem (22) has a unique weak solution.

let w be the solution of the problem, in this case we can consider the function $B(w(\cdot))$, since B is k-lipschitzian and $B(0) = 0$, it is clear that $B(w(\cdot)) \in L^2(0, T; L^2(\Omega))$ for all $T > 0$.
Since $\mathcal{A} = \partial\varphi$, Theorem 3.6 of
Brézis, H. Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50). North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.
yields that the w is, in fact, a strong solution in $[0, T]$ for all $T > 0$.

Asymptotic behavior

Suppose that $\phi(x,t)$ is the unique strong solution of Problem

$$\begin{array}{l} \partial_t \phi(x,t) = -A^*(A(f(\phi(x,t))) - m) + \beta \triangle \phi(x,t), \\ (\partial_\nu \phi(x,t) - r\phi(x,t))\,|_{\partial\Omega} = 0, \\ \phi(x,0) = \phi_0(x), \end{array}$$

with $\phi_0 \in L^2(\Omega)$, next we are going to study under what conditions the limit $\lim_{t\to\infty} \phi(x,t)$ exists for $r > 0$ and $\beta > 0$.

Asymptotic behavior

Suppose that $\phi(x,t)$ is the unique strong solution of Problem

$$\begin{array}{l}\partial_t\phi(x,t) = -A^*(A(f(\phi(x,t))) - m) + \beta\triangle\phi(x,t),\\ (\partial_\nu\phi(x,t) - r\phi(x,t))\,|_{\partial\Omega} = 0,\\ \phi(x,0) = \phi_0(x),\end{array}$$

with $\phi_0 \in L^2(\Omega)$, next we are going to study under what conditions the limit $\lim_{t\to\infty}\phi(x,t)$ exists for $r > 0$ and $\beta > 0$.

Theorem

Let Ω an open bounded subset of $\mathbb{R}^2$ with its boundary $\partial\Omega$ smooth. The operator $\mathcal{B} := -\beta\Delta + A^*Af$ is m-accretive whenever β is large enough.

Proof

1. It is well known that under this boundary condition Δ satisfies:

$$\langle -\Delta(u), u\rangle \geq \lambda_0 \|u\|_2^2 \ \text{ for all } \ u \in D(\mathcal{A}) \tag{24}$$

where $\lambda_0 > 0$ is the smallest eigenvalue of $-\Delta$ in $D(\mathcal{A})$.

Proof

1. It is well known that under this boundary condition Δ satisfies:

$$\langle -\Delta(u), u\rangle \geq \lambda_0 \|u\|_2^2 \ \text{ for all } \ u \in D(\mathcal{A}) \tag{24}$$

where $\lambda_0 > 0$ is the smallest eigenvalue of $-\Delta$ in $D(\mathcal{A})$.

2. This allows us to prove that $\mathcal{B}$ is ϕ-strongly accretive with $\phi(t) = (\beta\lambda_0 - k)t$.

Proof

1. It is well known that under this boundary condition Δ satisfies:

$$\langle -\Delta(u), u\rangle \geq \lambda_0 \|u\|_2^2 \quad \text{for all} \quad u \in D(\mathcal{A}) \tag{24}$$

   where $\lambda_0 > 0$ is the smallest eigenvalue of $-\Delta$ in $D(\mathcal{A})$.
2. This allows us to prove that $\mathcal{B}$ is ϕ-strongly accretive with $\phi(t) = (\beta\lambda_0 - k)t$.
3. $-\beta\Delta : D(\mathcal{A}) \to L^2(\Omega)$ is m-accretive, by Inequality (24), we derive that ψ-expansive with $\psi(t) = \beta\lambda_0 t$.

Proof

1. It is well known that under this boundary condition Δ satisfies:

$$\langle -\Delta(u), u\rangle \geq \lambda_0 \|u\|_2^2 \;\; \text{for all} \;\; u \in D(\mathcal{A}) \tag{24}$$

   where $\lambda_0 > 0$ is the smallest eigenvalue of $-\Delta$ in $D(\mathcal{A})$.
2. This allows us to prove that $\mathcal{B}$ is ϕ-strongly accretive with $\phi(t) = (\beta\lambda_0 - k)t$.
3. $-\beta\Delta : D(\mathcal{A}) \to L^2(\Omega)$ is *m*-accretive, by Inequality (24), we derive that ψ-expansive with $\psi(t) = \beta\lambda_0 t$.
4. we obtain that the operator $-\beta\Delta : D(\mathcal{A}) \to L^2(\Omega)$ is bijective.

Proof

1. It is well known that under this boundary condition Δ satisfies:
$$\langle -\Delta(u), u\rangle \geq \lambda_0 \|u\|_2^2 \quad \text{for all} \quad u \in D(\mathcal{A}) \tag{24}$$
where $\lambda_0 > 0$ is the smallest eigenvalue of $-\Delta$ in $D(\mathcal{A})$.
2. This allows us to prove that $\mathcal{B}$ is ϕ-strongly accretive with $\phi(t) = (\beta\lambda_0 - k)t$.
3. $-\beta\Delta : D(\mathcal{A}) \to L^2(\Omega)$ is m-accretive, by Inequality (24), we derive that ψ-expansive with $\psi(t) = \beta\lambda_0 t$.
4. we obtain that the operator $-\beta\Delta : D(\mathcal{A}) \to L^2(\Omega)$ is bijective.
5. Let $Q : L^2(\Omega) \to D(\mathcal{A})$ be the inverse operator of $-\beta\Delta : D(\mathcal{A}) \to L^2(\Omega)$, from inequality (24) it is clear that Q is continuous.

- To prove that $\mathcal{B}$ is m-accretive, we have to see that given $h \in L^2(\Omega)$ there exists $u \in D(\mathcal{A})$ such that $u + \mathcal{B}(u) = h$. This means that we have to solve the equation

$$u = \beta \Delta u - B(u) + h. \tag{25}$$

- To prove that $\mathcal{B}$ is m-accretive, we have to see that given $h \in L^2(\Omega)$ there exists $u \in D(\mathcal{A})$ such that $u + \mathcal{B}(u) = h$. This means that we have to solve the equation

$$u = \beta \Delta u - B(u) + h. \tag{25}$$

- In order to find a solution of Problem (25) it will be enough to show that the operator $K : L^2(\Omega) \to L^2(\Omega)$ defined by

$$K(u) = -Q(u) + B(Q(u)) - h$$

has a fixed point. Since, in this case, if u is a fixed point of K, then $v = Q(u)$ is a solution of Problem 25.

- To prove that $\mathcal{B}$ is m-accretive, we have to see that given $h \in L^2(\Omega)$ there exists $u \in D(\mathcal{A})$ such that $u + \mathcal{B}(u) = h$. This means that we have to solve the equation

$$u = \beta \Delta u - B(u) + h. \tag{25}$$

- In order to find a solution of Problem (25) it will be enough to show that the operator $K : L^2(\Omega) \to L^2(\Omega)$ defined by

$$K(u) = -Q(u) + B(Q(u)) - h$$

  has a fixed point. Since, in this case, if u is a fixed point of K, then $v = Q(u)$ is a solution of Problem 25.
- If we take β such that $\frac{1+\|A^*A\|}{\beta\lambda_0} < 1$ we achieve the result.

Theorem

Let $\beta\lambda_0 > \|A^*A\| + 1$. *If* $u : [0, \infty[\to L^2(\Omega)$ *is the strong solution of Problem*

$$
\begin{aligned}
&\partial_t \phi(x,t) = -A^*(A(f(\phi(x,t))) - m) + \beta \triangle \phi(x,t),\\
&(\partial_\nu \phi(x,t) - r\phi(x,t))\,|_{\partial\Omega} = 0,\\
&\phi(x,0) = \phi_0(x),
\end{aligned}
$$

with initial data $\phi_0 \in L^2(\Omega)$. *Then*

1. *there exists a unique* $\Phi \in \{u \in W^{2,2}(\Omega) : \ \partial_\nu u = ru \ a.\ e.\ on\ \partial\Omega\}$ *such that* $g = -\beta\Delta\Phi + A^*A(f(\Phi))$,
2. $u(t) \to \Phi$ *as* $t \to \infty$.

Proof

(a) *if* $\beta\lambda_0 > \|A^*A\| + 1$ *it is clear that* $\mathcal{B}$ *is* ψ*-strongly accretive*

Proof

(a) *if* $\beta\lambda_0 > \|A^*A\| + 1$ *it is clear that* $\mathcal{B}$ *is* ψ*-strongly accretive*

(b) *the operator* $\mathcal{H} = \mathcal{B} - g$ *is m-*ϕ*-accretive at zero.*

Proof

(a) *if* $\beta\lambda_0 > \|A^*A\| + 1$ *it is clear that* $\mathcal{B}$ *is* ψ*-strongly accretive*

(b) *the operator* $\mathcal{H} = \mathcal{B} - g$ *is m-*ϕ*-accretive at zero.*

(c) *There exists a unique* $\Phi \in D(\mathcal{B})$ *such that* $\mathcal{H}(\Phi) = 0$.

Proof

(a) *if* $\beta\lambda_0 > \|A^*A\| + 1$ *it is clear that* $\mathcal{B}$ *is* ψ*-strongly accretive*

(b) *the operator* $\mathcal{H} = \mathcal{B} - g$ *is m-*ϕ*-accretive at zero.*

(c) *There exists a unique* $\Phi \in D(\mathcal{B})$ *such that* $\mathcal{H}(\Phi) = 0$.

(d) *The strong solution is given by*

$$u(t) = \lim_{n\to\infty} (I + \frac{t}{n}\mathcal{H})^{-n}(\phi_0).$$

Proof

(a) *if* $\beta\lambda_0 > \|A^*A\| + 1$ *it is clear that* $\mathcal{B}$ *is* ψ*-strongly accretive*

(b) *the operator* $\mathcal{H} = \mathcal{B} - g$ *is m-*ϕ*-accretive at zero.*

(c) *There exists a unique* $\Phi \in D(\mathcal{B})$ *such that* $\mathcal{H}(\Phi) = 0$.

(d) *The strong solution is given by*

$$u(t) = \lim_{n\to\infty} (I + \frac{t}{n}\mathcal{H})^{-n}(\phi_0).$$

(e) $u(t) \to \Phi$ *as* $t \to \infty$.

Thank you