On a sufficient condition for equality of two maximal monotone operators

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Outline

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2 Examples: $\partial_\varepsilon f$ and T^ε

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5 Coincidence results based on enlargements

Inclusion Problem

$T : X \rightrightarrows X^*$ maximal monotone.

Problem (P_0): Find $x \in X$ such that

$$0 \in T(x)$$

Equivalently,

$$(x, 0) \in G(T)$$

Why we use enlargements

• Due to errors, instead of exact $G(T)$, we may know an outer approximation $G' \supset G(T)$.

• Being point-to-set, T is not continuous, so we approximate $G(T)$ by a bigger set $G' \supset G(T)$, where G' has better continuity properties.

Enlargements of T

Fix $T : X \rightrightarrows X^*$ maximal monotone.

$$\begin{array}{rcl} E : \mathbb{R}_+ \times X & \rightrightarrows & X^* \\ (\varepsilon, x) & \mapsto & E(\varepsilon, x) \end{array}$$

is a nondecreasing enlargement when

$$(\ e\ ) \quad T(x) \subset E(\varepsilon, x) \quad \forall\, x,\ \forall \varepsilon \geq 0$$

$$(nd) \quad E(\varepsilon', x) \subset E(\varepsilon, x) \quad \forall\, 0 \leq \varepsilon' \leq \varepsilon$$

Graph of E, $G(E) := \{(x, v, \varepsilon) \ : \ v \in E(\varepsilon, x)\}$

$$(\ e \ ) \qquad \longleftrightarrow \quad G(E) \supseteq G(T) \times \{0\}$$

$$(nd) \qquad \longleftrightarrow \quad G(E) \quad \text{epigraphical structure!}$$

$$(closed) \quad \longleftrightarrow \quad G(E) \text{ closed (allow to take lim!)}$$

Need extra requirements on E for

- having better knowledge of T.
- defining suitable approximations of (P_0).
- $E(\cdot, \cdot)$ is inner-semicontinuous.

Transportation Formula: tf

Take $v^1 \in E(\varepsilon_1, x^1)$, $v^2 \in E(\varepsilon_2, x^2)$
$\alpha_1, \alpha_2 \geq 0, \quad \alpha_1 + \alpha_2 = 1$, define:

$$\bar{x} := \alpha_1 x^1 + \alpha_2 x^2,$$

$$\bar{v} := \alpha_1 v^1 + \alpha_2 v^2,$$

$$\bar{\varepsilon} := \alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + \alpha_1 \alpha_2 \langle v^1 - v^2, x^1 - x^2 \rangle.$$

Then $\boxed{\bar{\varepsilon} \geq 0 \text{ and } \bar{v} \in E(\bar{\varepsilon}, \bar{x}).}$

The Family $Enl(T)$

$Enl(T)$ family of closed E: $\mathbb{R}_+ \times X \rightrightarrows X^*$,
verifying: $(\,e\,) + (nd) \;+ (tf)$

- $T(x) \subseteq E(\varepsilon, x), \quad \forall x, \varepsilon \geq 0$
- $E(\varepsilon, x) \subseteq E(\varepsilon', x), \quad \forall\, x,\, 0 \leq \varepsilon \leq \varepsilon'.$
- transportation formula

Svaiter, 2000.

The case $T = \partial f$

X real Banach space, X^* its dual.

$f : X \to \mathbb{R} \cup \{+\infty\}$ proper, convex, lsc.

$\partial f : X \rightrightarrows X^*$maximal monotone

$$\partial f(x) := \{v : f(y) \geq f(x) + \langle v, y - x \rangle \quad \forall y\}$$

$$\partial_{\varepsilon} f(x) := \{v : f(y) \geq f(x) + \langle v, y - x \rangle - \varepsilon \quad \forall y\}$$

Definition: Brøndsted & Rockafellar, 1965.

$E(\varepsilon, x) := \partial_{\varepsilon} f(x)$ satisf. (e) and (nd)

Properties of the ε-subdifferential

- Brøndsted & Rockafellar property (1965):

For $v_\varepsilon \in \partial_\varepsilon f(x_\varepsilon)$, $\rho > 0 \quad \exists\, v \in \partial f(x)$

$$\|v - v_\varepsilon\| \leq \varepsilon/\rho \qquad \|x - x_\varepsilon\| \leq \rho.$$

- Transportation Formula (Lemarechal, 1980).

- Lipschitz Continuity:

$$h(\partial_{\varepsilon} f(x), \partial_{\varepsilon'} f(x')) \leq \frac{k}{\min\{\varepsilon, \varepsilon'\}}(\|x - x'\| + |\varepsilon - \varepsilon'|)$$

(Nurminski, 1977, Hiriart-Urruty, 1980).

Generic T

$T : X \rightrightarrows X^*$ max. mon., $\varepsilon \geq 0$

$$\frac{v \in T(x) \quad \longleftrightarrow \quad \langle v - u, x - y \rangle \geq 0 \quad \forall (y,u) \in G(T)}{v \in B_T(\varepsilon, x) \longleftrightarrow \langle v - u, x - y \rangle \geq -\varepsilon \quad \forall (y,u) \in G(T)}$$

1996- B., Iusem, Svaiter. (dim$< \infty$)

1998- B., Sagastizábal, Svaiter. (Hilbert)

1999- B., Svaiter. (Banach)

Properties of B_T

- Brøndsted & Rockafellar property (Reflexive Banach).
- Transportation Formula.
- Lipschitz Continuity.

1999- B., Svaiter.

Properties of $E(\varepsilon, \cdot)$

- Brøndsted & Rockafellar property (Reflexive Banach).
- Lipschitz Continuity.
- Local Boundedness on $int(DomT)$.

Svaiter, 2000.

Convexity $\rightarrow$ Monotonicity

X real Banach space, X^* its dual.

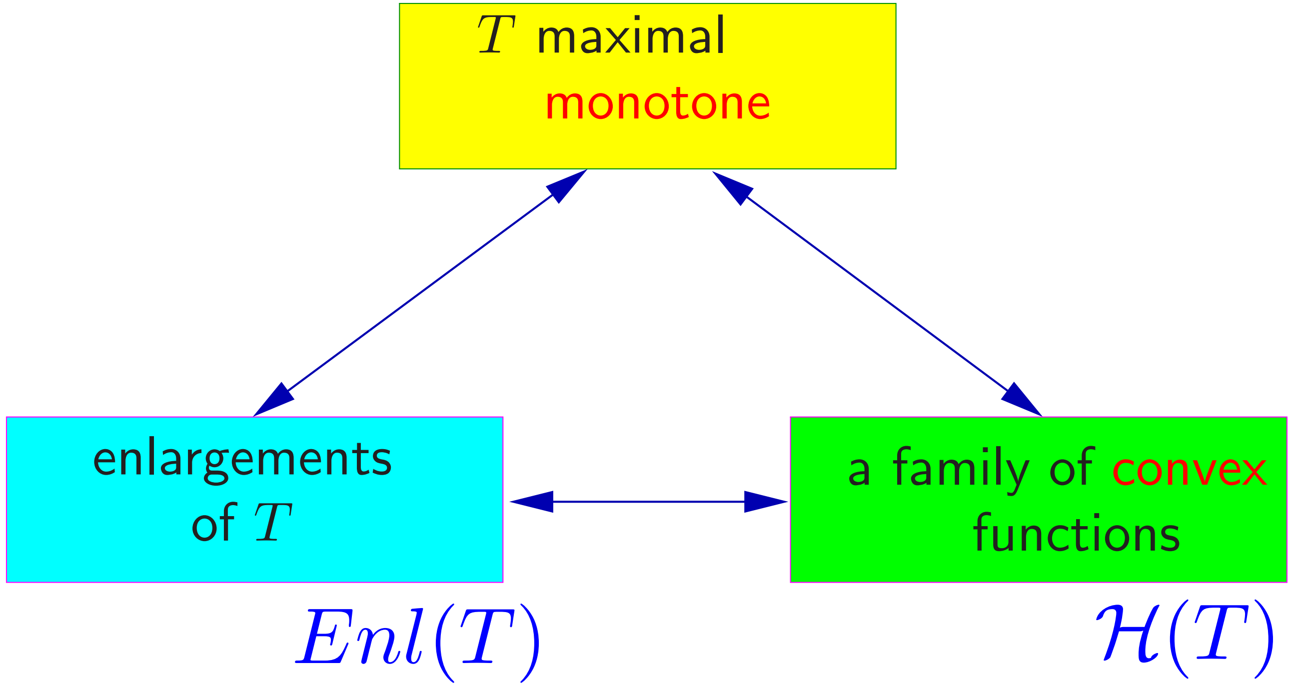
$$f : X \to \mathbb{R} \cup \{+\infty\}, \qquad \partial f : X \rightrightarrows X^*$$

f convex lsc	$\Longleftrightarrow$	∂f maximal monotone

$\Rightarrow$: Rockafellar, 1970

$\Leftarrow$: Correa, Jofré, Thibault, 1994

Monotonicity $\longrightarrow$ Convexity



$Enl(T)$ **& Convexity**

$$G(E) \;:=\; \{(x, v, \varepsilon) \;:\; v \in E(\varepsilon, x)\}$$

$$\searrow$$

$$\widetilde{G}(E) \;:=\; \{(x, v, \varepsilon + \langle x, v \rangle) \;:\; v \in E(\varepsilon, x)\}$$

$E \in Enl(T) \iff \widetilde{G}(E)$ { epigraph of a lsc convex function on $X \times X^*$ }

Define h_E so that $\widetilde{G}(E) = Epigraph(h_E)$

Properties of h_E

$$E \in Enl(T) \begin{cases} h_E \text{ convex lsc} \\ h_E(x,v) \geq \langle x,v \rangle \quad \forall x,\ v \\ h_E(x,v) = \langle x,v \rangle \quad \forall (x,v) \in G(T). \end{cases}$$

Family $\mathcal{H}(T)$

h: $X \times X^* \to \mathbb{R} \cup \{+\infty\}$ belongs to $\mathcal{H}(T)$ iff:

h convex lsc

$$h(x,v) \geq \langle x,v \rangle \quad \forall x,\ v$$

$$h(x,v) = \langle x,v \rangle \quad \forall (x,v) \in G(T).$$

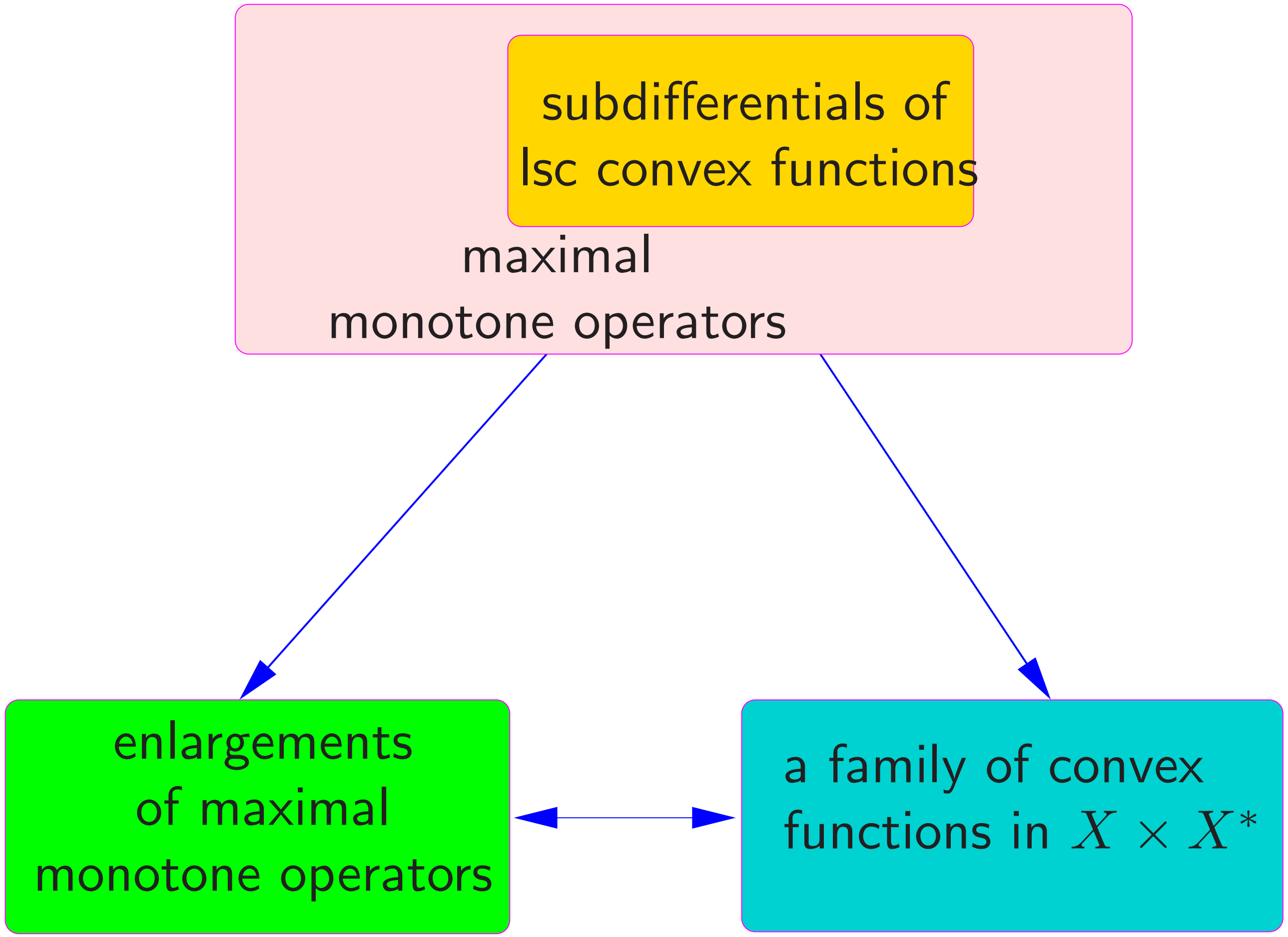
Studied for the first time by S. Fitzpatrick, 1988.

$$Enl(T) \leftrightarrow \mathcal{H}(T)$$

$$\begin{array}{ccc} Enl(T) & \longrightarrow & \mathcal{H}(T) \\ E & \overset{h_{(\cdot)}}{\longmapsto} & h_E \\ & & \\ L^h & \overset{L^{(\cdot)}}{\longleftarrow} & h \end{array}$$

where:

$$L^h(\varepsilon, x) := \{v \in X^* \ : \ h(x,v) \leq \varepsilon + \langle x, v \rangle\}$$


subdifferentials of
lsc convex functions
maximal
monotone operators
enlargements
of maximal
monotone operators
a family of convex
functions in $X \times X^*$

Examples of h_E

- $T = \partial f$, $E(\varepsilon, x) = \partial_\varepsilon f(x)$.

$$E \in Enl(\partial f), \quad h_E \in \mathcal{H}(\partial f).$$

$$h_E(x, v) = f(x) + f^*(v).$$

Fenchel-Young Inequality

$$f(x) + f^*(v) \geq \langle x, v \rangle \quad \forall x,\ v,$$

$$f(x) + f^*(v) = \langle x, v \rangle \quad \forall (x, v) \in G(\partial f).$$

Examples of h_E

Brezis-Haraux function, 1976.

$$\beta_T(x,v) := \sup\{\langle u-v, x-y\rangle \ : \ (y,u) \in G(T)\}$$

- T arbitrary max. mon.

$$B_T \in Enl(T), \quad h_{B_T} \in \mathcal{H}(T)$$

$$h_{B_T}(x,v) = \beta_T(x,v) + \langle x, v\rangle.$$

h_{B_T} was studied by S. Fitzpatrick in 1988, now called the *Fitzpatrick function*

Characterization of $\mathcal{H}(\partial f)$

h is separable if $h(x,v) = f(x) + g(v)$

f convex lsc	$\Longrightarrow$	$\mathcal{H}(\partial f)$ has separable member: $h_E(x,v) = f(x) + f^*(v)$

Conversely	$\mathcal{H}(T)$ has sep. member $h(x,v) = f(x) + g(v)$	$\Longrightarrow$	$g = f^*$ $T = \partial f$

B., Svaiter, 2002.

The sublinear case

Fact 1: X arbitrary Banach and f everywhere defined, lsc convex sublinear, then $\mathcal{H}(\partial f)$, (and $Enl(\partial f)$) has only one element.

[Reflexive case: Penot,2004]

[Arbitrary Banach: B., Fitzpatrick, 2004]

Corollary In case $f = \|\cdot\|$, then

$$\mathcal{H}(\partial f) = \{\|\cdot\| + \delta_{B^*}\}$$

($B^* \subset X^*$ unit ball)

T monotone only

Consider T^μ: all points monotonically related to T, i.e.

$$G(T^\mu) := \{(x, x^*) : \langle x - y, x^* - y^* \rangle \geq 0 \,\forall (y, y^*) \in G(T)\}$$

Martínez-Legaz, Svaiter, 2005.

Convex-like Sets

Y vector space, $A \subseteq Y$

A *convex-like* $\iff$ $x, y \in A$, $x \neq y$, $\implies (x, y) \cap A \neq \emptyset$

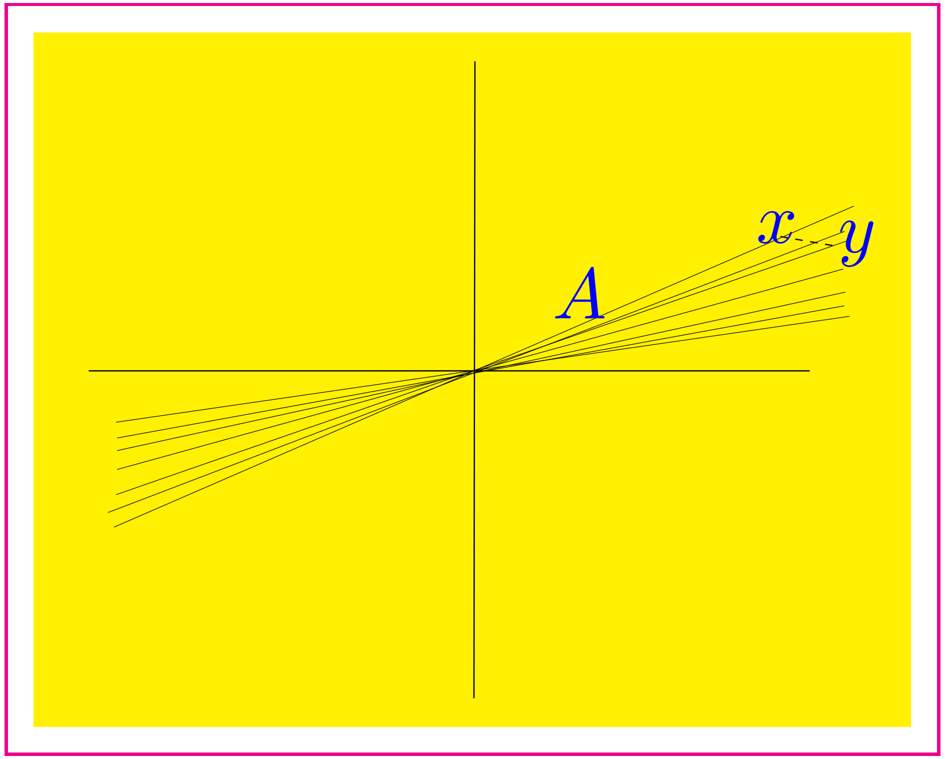


Figure 1: An example of a convex-like set A.

Coincidence Result 1

$$]x, y_{+\infty}[:= \{x + t(y - x) \ : \ t \geq 0\}$$

Theo 1 $S, T : X \rightrightarrows X^*$ monotone. Assume that

$$D(S) \ni x \neq y \in D(T) \Longrightarrow S(]x, y_{+\infty}[) \cap T(]y, x_{+\infty}[) \neq \emptyset$$

Then $S \subseteq T^\mu$ and $T \subseteq S^\mu$. In particular, if T maximal, then $S \subseteq T$, and if both S and T maximal then $S = T$

Corollary 1

$T, S : X \rightrightarrows X^*$ monotone and T maximal. If $D(T) = D(S) =: D$ convex-like and $T(x) \cap S(x) \neq \emptyset \Longrightarrow S \subset T$. In particular, if S also maximal, then $T = S$

Corollary 2

$f, g : X \to \mathbb{R}_{+\infty}$ lsc convex. For the statements:

(i) If $D(\partial f) \ni x \neq y \in D(\partial g)$ then

$$\partial f(]x, y_{+\infty}[) \cap \partial g(]y, x_{+\infty}[) \neq \emptyset$$

(ii) $f(\cdot) - g(\cdot) = c$ for some $c \in \mathbb{R}$

(iii) $D(\partial f) = D(\partial g)$ and $\partial f(x) \cap \partial g(x) \neq \emptyset \, \forall x$

We have that $(i) \implies (ii) \implies (iii)$

$D(\partial f), D(\partial g)$ convex-like $\implies$ all equivalent

Coincidence Results based on enlargements

$T, S : X \rightrightarrows X^*$ maximal monotone with $D(T), D(S)$ open
$E_T, E_S : X \times \mathbb{R}_+ \rightrightarrows X^*$ closed enlargements of T, S
For the statements:

(i) Whenever $D(S) \ni x \neq y \in D(T)$, then $\exists\, u \in]x, y_{+\infty}[\cap D(T)$ and $v \in]y, x_{+\infty}[\cap D(S)$ s.t. $E_T(u, \varepsilon) \cap E_S(v, \varepsilon) \neq \emptyset$, $\forall\, \varepsilon$

(ii) $T = S$

(iii) $D(T) = D(S) =: D$ and $E_T(x, \varepsilon) \cap E_S(x, \varepsilon) \neq \emptyset$ for every $x \in D$, $\varepsilon > 0$

We have $(i) \implies (ii) \implies (iii)$
If $D(T)$ and $D(S)$ are convex-like, all are equivalent

Corollary

f, g lsc convex whose subdifferentials have same open convex-like domain D. TFSAE:

(i) $\partial_\varepsilon f(x) \cap \partial_\varepsilon g(x) \neq \emptyset,\ \forall x \in D, \varepsilon > 0$

(ii) $\exists\, c \in \mathbb{R}$ such that $f(\cdot) - g(\cdot) = c$

(iii) $(\partial f)^\varepsilon(x) \cap (\partial g)^\varepsilon(x) \neq \emptyset,\ \forall x \in D, \varepsilon > 0$