On the lower semicontinuity of the feasible set mapping in optimization

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Outline

- Parameter spaces

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- Some desirable stability properties

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where:

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- $C \subset X$.
- $f, f_t : X \to \mathbb{R} \cup \{\pm\infty\} \ \forall t \in T$.

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- The *parameter space* Θ is the set of perturbed data

$$\sigma_1 = \{f_t^1, t \in T;\ C_1\}$$

"similar" to σ.

Parameter spaces

We take seven parameter spaces as milestones:

- $\sigma_1 \in \Theta_1$ if $f_t^1 : X \to \mathbb{R} \cup \{+\infty\}$ $\forall t \in T$ and $\emptyset \neq C_1 \subset X$.

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- $\sigma_1 \in \Theta_4$ if $\sigma_1 \in \Theta_2$, all the local mininima of g^1 are global, and $C_1 = X$.

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- Then

$$\begin{array}{ccccccc} \Theta_4 & \subset & \Theta_3 & \subset & \Theta_2 & \subset & \Theta_1 \\ \cup & & & & \cup & & \\ \Theta_7 & \subset & \Theta_6 & \subset & \Theta_5 & & \end{array}$$

Some desirable stability properties

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Let $\Theta \subset \Theta_1$, Θ_1 equipped with some topology.
The *feasible set mapping* is $\mathcal{F} : \Theta \rightrightarrows X$ such that

$$\mathcal{F}(\sigma_1) = \{x \in X : f_t^1(x) \leq 0 \ \forall t \in T; \ x \in C_1\}$$

and its *domain*

$$\operatorname{dom} \mathcal{F} = \{\sigma_1 \in \Theta : \mathcal{F}(\sigma_1) \neq \emptyset\}.$$

- $\mathcal{F}$ is *closed* at $\sigma \in \Theta$ if $\forall \{\sigma_\delta\} \subset \Theta$ and $\forall \{x_\delta\} \subset X$ such that $x_\delta \in \mathcal{F}(\sigma_\delta)$ $\forall \delta \in \Delta$, with $\sigma_\delta \to \sigma$ and $x_\delta \to x$, one has

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$$x \in \mathcal{F}(\sigma).$$
- $\mathcal{F}$ is *lsc* at $\sigma \in \Theta$ if $\forall$ open set $W \subset X$ such that $W \cap \mathcal{F}(\sigma) \neq \emptyset$, $\exists V \subset \Theta$ open, $\sigma \in V$, such that
$$W \cap \mathcal{F}(\sigma_1) \neq \emptyset \ \forall \sigma_1 \in V.$$

Some desirable stability properties

- σ is *Tuy regular* if

$$\sigma_1 = \{f_t - u_t, t \in T;\ C_1\} \in \operatorname{dom} \mathcal{F}, \text{ with } u_t \in \mathbb{R}\ \forall t \in T,$$

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$\forall \sigma_1 \in \Theta$ "sufficiently close" to σ.

- σ satisfies the *SS condition* if $\exists \bar{x} \in C$ and $\exists \rho > 0$ such that

$$f_t(\bar{x}) < -\rho\ \forall t \in T$$

(i.e., g satisfies Slater condition). We denote by F_{SS} the set of all SS points of σ.

Some desirable stability properties

The *inferior and superior limits* of the net of sets $\mathcal{F}(\sigma_\delta)$, $\delta \in \Delta$, in *Kuratowski-Painlevé sense, are*

$$\operatorname*{Li}_{\delta} \mathcal{F}(\sigma_\delta) = \left\{ x \in X \;\middle|\; \begin{array}{l} \forall U \text{ nghbd of } x\ \exists \delta \in \Delta \text{ such} \\ \text{that } U \cap \mathcal{F}(\sigma_{\delta'}) \neq \emptyset \\ \forall \delta' \in \Delta \text{ such that } \delta \preceq \delta' \end{array} \right\},$$

$$\operatorname*{Ls}_{\delta} \mathcal{F}(\sigma_\delta) = \left\{ x \in X \;\middle|\; \begin{array}{l} \forall U \text{ nghbd of } x \text{ and } \forall \delta \in \Delta \\ \exists \delta' \in \Delta \text{ such that } \delta \preceq \delta' \\ \text{and } U \cap \mathcal{F}(\sigma_{\delta'}) \neq \emptyset \end{array} \right\}.$$

- We say that $\sigma \in \Theta$ is *Kuratowski-Painlevé stable* if

$$\operatorname*{Li}_{\delta} \mathcal{F}(\sigma_\delta) = \operatorname*{Ls}_{\delta} \mathcal{F}(\sigma_\delta) = \mathcal{F}(\sigma)$$

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- Robinson regularity will be defined later (it requires X to be a normed space).

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- Given $f, h \in \mathcal{V}_1 := \left(\mathbb{R} \cup \{+\infty\}\right)^X$, we define

$$d_k(f, h) := \sup_{x \in B_k} |f(x) - h(x)|, \ k \in \mathbb{N},$$

and

$$d(f, h) := \sum_{k=1}^{+\infty} 2^{-k} \min\{1, d_k(f, h)\},$$

where $(+\infty) - (+\infty) = 0$, $|-\infty| = +\infty$, $|+\infty| = +\infty$.

Topology on the parameter space

Consider $\mathcal{V}_1 = (\mathbb{R} \cup \{+\infty\})^X$ and the nested sets

$$\begin{aligned}
\mathcal{V}_2 &:= \{f \in \mathcal{V}_1 : f \text{ is lsc}\} \\
\mathcal{V}_j &= \{f \in \mathcal{V}_2 : \text{the loc. min. of } f \text{ are global}\}, j = 3, 4 \\
\mathcal{V}_5 &:= \{f \in \mathcal{V}_2 : f \text{ is convex}\} \\
\mathcal{V}_6 &:= \{f \in \mathcal{V}_5 : f : X \to \mathbb{R}\} \\
\mathcal{V}_7 &:= X^*
\end{aligned}$$

- The improper function $\{+\infty\}^X$ (with constant value $+\infty$) is an accumulation point of $\mathcal{V}_j$, $j = 1, ..., 5$, because $\{+\infty\}^X = \lim_k \delta_{\{x_k\}}$, where $x_k \in X \diagdown B_k$ for all $k \in \mathbb{N}$.

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- The topology of $\mathcal{V}_7$ describes the uniform convergence of the continuous linear functionals on B.

Topology on the parameter space

Theorem

$(\mathcal{V}_j, d)$ *is a metric space for all* j *which is complete if* $j = 1, 2, 5, 6, 7$.

The next example shows that $\mathcal{V}_4 = \mathcal{V}_3$ is nonclosed.

Example 1:

- Let $X = \mathbb{R}$ and $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{V}_3$ be such that

$$f_n(x) = \begin{cases} |x|, & \text{if } x \leq 1 \\ \frac{x+n}{n+1}, & \text{if } x \in \left]1, \frac{2n+1}{n}\right[\\ x-1, & \text{if } x \geq \frac{2n+1}{n}. \end{cases}$$

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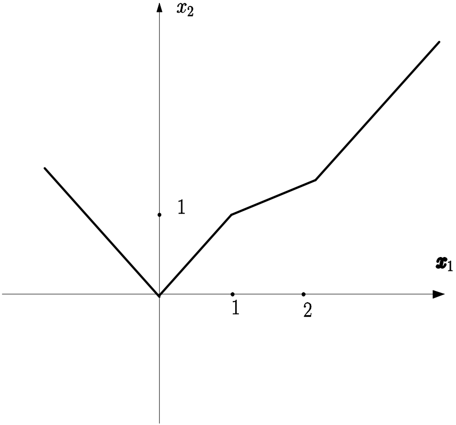
- Then $f_n \to f \notin \mathcal{V}_3$, with

$$f(x) = \begin{cases} |x|, & \text{if } x \leq 1 \\ 1, & \text{if } x \in \left]1, 2\right[\\ x-1, & \text{if } x \geq 2. \end{cases}$$

Example 1 (continues):



gph f_n

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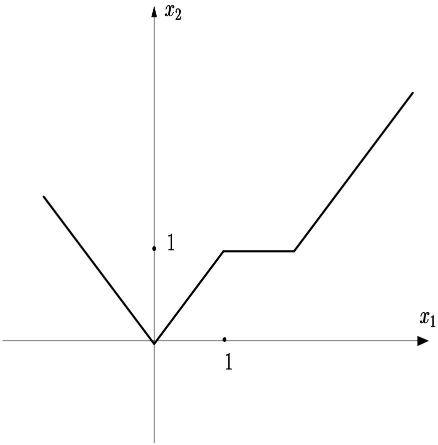

gph f

Topology on the parameter space

- Given $\sigma, \sigma_1 \in \Theta_1$, we define

$$\mathbf{d}(\sigma, \sigma_1) = \begin{cases} \max\{\sup_{t \in T} d(f_t, f_t^1), d(\delta_C, \delta_{C_1})\}, & \text{if } T \neq \emptyset \\ d(\delta_C, \delta_{C_1}), & \text{if } T = \emptyset. \end{cases}$$

Theorem

$(\Theta, \mathbf{d})$ is a metric space $\forall \Theta \subset \Theta_1$.
$(\Theta, \mathbf{d})$ is complete if Θ is a closed subset of Θ_1.
Θ_j is closed in Θ_1, $j = 2, 5, 6, 7$.

Example 1 (continues)**:**
Let

$$\sigma_n = \{f_n; \mathbb{R}\} \in \Theta_4 \subset \Theta_3, n \in \mathbb{N}.$$

Since

$$\sigma_n \to \sigma = \{f; \mathbb{R}\} \notin \Theta_3,$$

Θ_3 and Θ_4 are nonclosed.

Theorem

$\mathcal{F}$ is closed everywhere if $\Theta \subset \Theta_2$.

Lower semicontinuity

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Otherwise, we are interested in the relationship between the following desirable stability properties at $\sigma \in \operatorname{dom} \mathcal{F}$:

- (i) $\mathcal{F}$ is lsc at σ.

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- (iii) σ is Tuy regular.

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- (iii) σ is Tuy regular.
- (iv) σ satisfies the SS condition.

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- (v) $\mathcal{F}(\sigma) = \operatorname{cl} F_{SS}$.

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- (iii) σ is Tuy regular.
- (iv) σ satisfies the SS condition.
- (v) $\mathcal{F}(\sigma) = \operatorname{cl} F_{SS}$.
- (vi) σ is Kuratowski-Painlevé stable.

Lower semicontinuity

- (vii) $0 \notin \operatorname{cl}\operatorname{conv}\left\{\left(\bigcup_{t\in T}\operatorname{epi} f_t^*\right) + \operatorname{epi}\delta_C^*\right\}$.
  (with the usual convention that $\emptyset + \operatorname{epi}\delta_C^* = \emptyset$.)

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  (with the usual convention that $\emptyset+\operatorname{epi}\delta_C^*=\emptyset$.)
- (viii) $0 \notin \operatorname{cl}\operatorname{conv}\left\{\left(\bigcup_{t\in T}\operatorname{epi} f_t^*\right)\cup\left[\operatorname{epi}\delta_C^*+(0,1)\right]\right\}$.

Lower semicontinuity

Theorem

Let $\sigma \in \operatorname{dom} \mathcal{F}$ *be such that* $T \neq \emptyset$ *and* $\Theta \subset \Theta_1$.
If $\sigma \in \Theta_1$, *then (i)⇒(ii)⇒(iii)⇒(iv).*
If $\sigma \in \Theta_3 \cup \Theta_5$, *then (i)-(vi) are equivalent to each other.*
If $\sigma \in \Theta_5$, *then (i)-(viii) are equivalent to each other.*

Corollary

Let $T \neq \emptyset$.
If $\Theta_7 \subset \Theta \subset \Theta_3$, *then (i)-(vi) are equivalent to each other* $\forall \sigma \in \operatorname{dom} \mathcal{F}$.
If $\Theta_7 \subset \Theta \subset \Theta_5$, *then (i)-(viii) are equivalent to each other* $\forall \sigma \in \operatorname{dom} \mathcal{F}$.

Lower semicontinuity

- If $\sigma = \{f_t, t \in T; C\} \in \Theta_6 \cap \operatorname{dom} \mathcal{F}$ and $C = X$ (e.g., $\sigma \in \Theta_7 \cap \operatorname{dom} \mathcal{F}$), then (viii)$\Rightarrow$ (vii).

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- The converse statement is true whenever -1 is a lower bound for some constraint function f_t.

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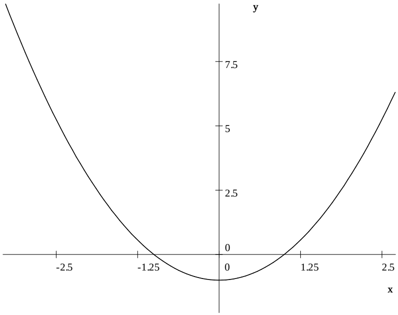
- The converse statement is true whenever -1 is a lower bound for some constraint function f_t.
- In Examples 2 and 3, where $|T| = 1$ and $\sigma \in \Theta_2 \diagdown (\Theta_3 \cup \Theta_5)$, (ii)-(iv), (vii) and (viii) hold whereas (i), (v), and (vi) fail.

Example 2:
Let $\sigma = \{f; C\} \in \Theta_2 \diagdown (\Theta_3 \cup \Theta_5)$ be such that $f(x) = x^2 - 1$ and $C = \{-1, 0, 1\} \subset \mathbb{R}$ (C nonconvex).



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Lower semicontinuity

Example 2 (continues)**:**

- We have $\mathcal{F}(\sigma) = \{-1, 0, 1\}$ and $\mathcal{F}(\sigma_n) = \{0\}$ for

$$\sigma_n = \left\{ f + \frac{1}{n}; C \right\}, n \in \mathbb{N},$$

  with $\sigma_n \to \sigma$.

  - $\mathcal{F}$ is not lsc at σ (take $W =]0, 2[$).

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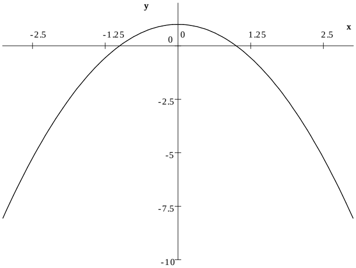
  - $\mathcal{F}$ is not lsc at σ (take $W =]0, 2[$).
  - $\operatorname{cl} F_{SS} = \{0\} \neq \mathcal{F}(\sigma)$.
  - $\operatorname{Li}_n \mathcal{F}(\sigma_n) = \operatorname{Ls}_n \mathcal{F}(\sigma_n) = \{0\} \neq \mathcal{F}(\sigma)$.

Example 3:

Let $\sigma = \{f; C\} \in \Theta_2 \diagdown (\Theta_3 \cup \Theta_5)$ be such that $f(x) = 1 - x^2$ and $C = [-1, +\infty[\subset \mathbb{R}$.



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- Let $\sigma_n = \left\{f + \frac{1}{n}; C\right\}$, $n \in \mathbb{N}$. Then

$$\mathcal{F}(\sigma_n) = \left[\sqrt{\frac{n+1}{n}}, +\infty\right[\ \forall n \in \mathbb{N},$$

and $\sigma_n \to \sigma$.

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- $\mathcal{F}(\sigma) = \{-1\} \cup [1, +\infty[\nsubseteq \operatorname{int} C$.
- Let $\sigma_n = \left\{f + \frac{1}{n}; C\right\}$, $n \in \mathbb{N}$. Then

$$\mathcal{F}(\sigma_n) = \left[\sqrt{\frac{n+1}{n}}, +\infty\right[\ \forall n \in \mathbb{N},$$

  and $\sigma_n \to \sigma$.
- $\mathcal{F}$ is not lsc at σ (take $W =]-2, 0[$).

Example 3 (continues):

- $\mathcal{F}(\sigma) = \{-1\} \cup [1, +\infty[\nsubseteq \operatorname{int} C$.
- Let $\sigma_n = \left\{f + \frac{1}{n}; C\right\}$, $n \in \mathbb{N}$. Then

$$\mathcal{F}(\sigma_n) = \left[\sqrt{\frac{n+1}{n}}, +\infty\right[\ \forall n \in \mathbb{N},$$

  and $\sigma_n \to \sigma$.
- $\mathcal{F}$ is not lsc at σ (take $W =]-2, 0[$).
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- $\mathrm{cl}\, F_{SS} = [1, +\infty[\neq \mathcal{F}(\sigma)$.

Lower semicontinuity

- Assume that X is a normed space with associated distance δ. Let $\delta(x, \emptyset) = +\infty \ \forall x \in X$.

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- $\mathcal{F} : \Theta \rightrightarrows X$ is *Robinson regular* at $\sigma \in \Theta$ if $\forall \widetilde{x} \in \mathcal{F}(\sigma)$, $\exists \epsilon, \beta > 0$ such that

$$\begin{cases} \widetilde{x} \in C_1 \text{ and} \\ \delta(\widetilde{x}, \mathcal{F}(\sigma_1)) \leq \beta \max\left\{0, g^1(\widetilde{x})\right\}, \end{cases}$$

$\forall \sigma_1 \in \Theta$ such that $\mathbf{d}(\sigma, \sigma_1) < \epsilon$.

Theorem

Let $\sigma \in \operatorname{dom} \mathcal{F}$.
If $\mathcal{F}$ *is Robinson regular at* $\sigma \in \Theta_3 \cup \Theta_5$, *then* $\mathcal{F}$ *is lsc at* σ.
If $\mathcal{F}$ *is lsc at* σ *and* $\sigma \in \Theta_5$, *then* $\mathcal{F}$ *is Robinson regular at* σ.

Corollary

If $\Theta_7 \subset \Theta \subset \Theta_5$ *and* $\sigma \in \operatorname{dom} \mathcal{F}$, *then*

$$\mathcal{F} \text{ is Robinson regular at } \sigma \Leftrightarrow \mathcal{F} \text{ is lsc at } \sigma$$

- Convexity is also essential for this equivalence.

Example 4:

Let $\sigma = \{f; \mathbb{R}\} \in \Theta_3$, where $f(x) = -x^2$.

- Take $\widetilde{x} = 0 \in \mathcal{F}(\sigma)$ and $\sigma_n = \left\{f + \frac{1}{n}; \mathbb{R}\right\} \in \Theta_3 \ \forall n \in \mathbb{N}$. Obviously, $\sigma_n \longrightarrow \sigma$.

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- $\delta(\widetilde{x}, \mathcal{F}(\sigma_n)) = \frac{1}{\sqrt{n}}$ and $\max\{0, g^n(\widetilde{x})\} = \frac{1}{n}$, $\forall n \in \mathbb{N}$.

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- $\delta(\widetilde{x}, \mathcal{F}(\sigma_n)) = \frac{1}{\sqrt{n}}$ and $\max\{0, g^n(\widetilde{x})\} = \frac{1}{n}$, $\forall n \in \mathbb{N}$.
- Assuming that $\mathcal{F}$ is Robinson regular at σ, with constant $\beta > 0$, we must have $\frac{1}{\sqrt{n}} \leq \frac{\beta}{n}$ for n large enough (contradiction).

Bibliographical notes

Bibliographical notes

1970s	(P)	cont	top	Rob	Tuy
Robinson 75	LIP, X Banach			✓	
Daniel 75	LP, $\Theta = \Theta_7$			✓	
GrePier 75	SIP, $\Theta = \Theta_2$	✓			
Robinson 76	$\mathcal{C}^1$-IP, X Banach			✓	✓
Tuy 77	LIP, $\Theta = \Theta_7$, X lcHtvs				✓

Bibliographical notes

1980s	(P)	cont	top	Rob	Tuy
Brosowski 82	LSIP, $\Theta \subsetneq \Theta_7$	✓			
Fischer 83	LSIP, $\Theta \subsetneq \Theta_7$	✓			
Bank et al. 83	CP, $\Theta \subsetneq \Theta_6$	✓			
Brosowski 84	LSIP, $\Theta \subsetneq \Theta_7$	✓			
Todorov 85/86	LSIP, $\Theta \subsetneq \Theta_7$	✓			
GuJoRü 86	$\mathcal{C}^1$-MP, $\Theta \subsetneq \Theta_2$		✓		

Bibliographical notes

1990s	(P)	cont	top	Rob	Tuy
Helbig 90	LSIP, $\Theta \subsetneq \Theta_7$	✓			
JoTwiWe 92	$\mathcal{C}^1$-SIP, $\Theta \subsetneq \Theta_2$	✓	✓		
LuLu 94	LP, $\Theta = \Theta_7$			✓	
GoLóTo 96	LSIP, $\Theta = \Theta_7$	✓		✓	✓
GoLó 96	LSIP, $\Theta = \Theta_7$	✓	✓		
GoLóTo 97	LSIP, $\Theta = \Theta_7$	✓			
JoRü 98	$\mathcal{C}^1$-SIP, $\Theta \subsetneq \Theta_2$		✓		
GoLó 98	LSIP, $\Theta = \Theta_7$	✓	✓	✓	✓
LóMiTo 98	LIP, $\Theta = \Theta_7$, X lcHtvs	✓			✓

Main antecedents

In 2000-04	(P)	cont	top	Rob	Tuy
Hu 00	LSIP, $\Theta = \Theta_7$			✓	
MiMo 00	LIP, $\Theta = \Theta_7$, X lcHtvs	✓			
GoLóTo 01	LIP, $\Theta \subset \Theta_6$, $C_0 = \mathbb{R}_+^{(T)}$	✓			✓
LóVe 01	CSIP, $\Theta = \Theta_6$, $C_0 = \mathbb{R}^n$	✓		✓	✓
CáLóPa 02	LSIP, $\Theta = \Theta_7$	✓			

Bibliographical notes

2005-09	(P)	cont	top	Rob	Tuy
CáDoLóPa 05	LSIP, $\Theta \subsetneq \Theta_7$	✓		✓	
LóRuVe 05	min-type SIP, $\Theta \subsetneq \Theta_4$	✓		✓	✓
AmGo 06	LSIP, $\Theta_7 \subset \Theta \subset \Theta_6$	✓			
DiGoLóSo 07	CIP, $\Theta = \Theta_5$, X lcHtvs				✓
AmBoGo 08	LSIP, $\Theta_7 \subset \Theta \subset \Theta_6$	✓			
CáGóPa 08	LSIP, $\Theta = \Theta_7$	✓		✓	

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