Convex representations of monotone operators, surjectivity theorems and positive sets

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$X \neq \{0\}$ reflexive real Banach space, X^* its dual

$\langle \cdot, \cdot \rangle : X \times X^* \longrightarrow \mathbb{R}$ the duality pairing

$J = \partial \frac{1}{2} \|\cdot\|^2$, the duality mapping

For $x \in X, \quad J(x) = \left\{ x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}.$

THEOREM (Rockafellar, 1970). *Let $A : X \rightrightarrows X^*$ be a monotone operator. In order that A be maximal, it is necessary and sufficient that $R(A+J)$ be all of X^*.*

$$\begin{aligned} A &: X \rightrightarrows X^* \\ \varphi_A &: X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\} \\ \varphi_A(x, x^*) &= \langle x, x^* \rangle - \inf_{(y,y^*) \in Graph(A)} \langle x - y, x^* - y^* \rangle \end{aligned}$$

(Fitzpatrick, 1988)

If A is maximal monotone,

$$\begin{aligned} \varphi_A(x, x^*) &\geq \langle x, x^* \rangle \qquad \forall \ (x, x^*) \in X \times X^*, \\ \varphi_A(x, x^*) &= \langle x, x^* \rangle \qquad \Longleftrightarrow \qquad (x, x^*) \in Graph(A). \end{aligned}$$

$h : X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a *convex representation* of A if it is convex and l.s.c. and satisfies

$$\begin{aligned} h(x, x^*) &\geq \langle x, x^* \rangle \qquad \forall\ (x, x^*) \in X \times X^*, \\ h(x, x^*) &= \langle x, x^* \rangle \qquad \Longleftrightarrow \qquad (x, x^*) \in Graph(A). \end{aligned}$$

φ_A is the smallest convex representation of A.

If $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is convex and l.s.c. then

$$\begin{gathered} h : X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\} \\ h(x, x^*) = f(x) + f^*(x^*) \end{gathered}$$

is a convex representation of ∂f.

If $h : X \times X^* \longrightarrow \mathbb{R}$ is a convex representation of A then

$$\langle \cdot, \cdot \rangle \leq h \leq \langle \cdot, \cdot \rangle + \delta_{Graph(A)}.$$

$$\sigma_A = cl\ conv\left(\langle \cdot, \cdot \rangle + \delta_{Graph(A)}\right)$$

(Burachik-Svaiter, 2002)

$$\langle \cdot, \cdot \rangle \leq h \leq \sigma_A \leq \langle \cdot, \cdot \rangle + \delta_{Graph(A)}.$$

σ_A is the largest convex representation of A.

THEOREM (ML-Svaiter, 2005). *Let* $A : X \rightrightarrows X^*$. *Then*

$$A \text{ is monotone} \qquad \Longleftrightarrow \qquad \sigma_A \geq \langle \cdot, \cdot \rangle .$$

Proof of $\Longrightarrow$.
Let M be a maximal monotone extension of A.
$\sigma_A \geq \sigma_M \geq \varphi_M \geq \langle \cdot, \cdot \rangle$

Proof of $\Longleftarrow$.

Let (x, x^*), $(y, y^*) \in Graph(A)$.

$$\begin{aligned}\left\langle \frac{1}{2}(x+y), \frac{1}{2}(x^*+y^*) \right\rangle &\leq \sigma_A\left(\frac{1}{2}(x+y), \frac{1}{2}(x^*+y^*)\right) \\ &\leq \frac{1}{2}\left(\sigma_A(x,x^*) + \sigma_A(y,y^*)\right) \\ &= \frac{1}{2}\left(\langle x, x^*\rangle + \langle y, y^*\rangle\right)\end{aligned}$$

$$\langle x-y, x^*-y^* \rangle \geq 0$$

Example: $Graph(A) = \{(0,0)\} \qquad \varphi_A \equiv 0$

If A is maximal monotone,

$$\begin{aligned}\varphi_A(x,x^*) &= \langle x, x^* \rangle - \inf_{(y,y^*)\in A} \langle x-y, x^*-y^* \rangle \\ &= \sup_{(y,y^*)\in A} \{\langle x, y^*\rangle + \langle y, x^*\rangle - \langle y, y^*\rangle\} \\ &= (\langle \cdot, \cdot \rangle + \delta_A)^*(x^*, x) \\ &= \sigma_A^*(x^*, x) \qquad \forall\ (x,x^*) \in X \times X^*\end{aligned}$$

$$\sigma_A(x,x^*) = \varphi_A^*(x^*,x) \qquad \forall\ (x,x^*) \in X \times X^*$$

Examples (X Hilbert space, B its unit ball):

$$f : X \to X \qquad f(x) = \frac{1}{2}\|x\|^2$$

$$\partial f(x) = \{x\}$$

$$\varphi_{\partial f}(x, x^*) = \frac{1}{4}\|x + x^*\|^2$$

$$f(x) + f^*(x^*) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2$$

$$\sigma_{\partial f}(x, x) = \|x\|^2, \ \sigma_{\partial f}(x, x^*) = +\infty \text{ if } x \neq x^*$$

$$T : X \to X \text{ linear, continuous, } T^* = -T$$

$$\langle T(x), y\rangle + \langle x, T(y)\rangle = 0 \qquad \forall \ (x, y) \in X^2$$

$$\varphi_T = \delta_{Graph(T)} = \langle \cdot, \cdot \rangle + \delta_{Graph(T)} = \sigma_T$$

$$\emptyset \neq C \subseteq X, \text{ closed convex set} \qquad \delta_C : X \to \mathbb{R} \cup \{+\infty\}$$

$$\partial \delta_C(x) = \begin{cases} \{x^* \in X : \langle y - x, x^*\rangle \leq 0 \ \forall \ y \in C\} & \text{if } x \in C \\ \emptyset & \text{if } x \notin C \end{cases}$$

$$\varphi_{\partial \delta_C}(x, x^*) = \delta_C(x) + \delta_C^*(x^*)$$

$$\sigma_{\partial \delta_C}(x, x^*) = \delta_C(x) + \delta_C^*(x^*)$$

$$f : X \rightarrow X \qquad f(x) = \|x\|$$

$$\partial f(x) = \begin{cases} \left\{ \frac{x}{\|x\|} \right\} & \text{if } x \neq 0 \\ B & \text{if } x = 0 \end{cases}$$

$$\varphi_{\partial f}(x, x^*) = \|x\| + \delta_B(x^*) = f(x) + f^*(x^*)$$

$$\sigma_{\partial f}(x, x^*) = f(x) + f^*(x^*)$$

$f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ sublinear, l.s.c. $\Longrightarrow$ $\varphi_{\partial f} = \sigma_{\partial f}$

(Burachik-Fitzpatrick, 2005)

B satisfies the Brézis-Haraux condition if

$$\inf_{(y,y^*) \in Graph(B)} \langle x - y, x^* - y^* \rangle > -\infty \quad \forall\ (x, x^*) \in D(B) \times R(B).$$

φ_B is finite-valued $\Longrightarrow$ B satisfies the B-H condition

THEOREM. (Rockafellar, 1966). Suppose that $f, g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ are l.s.c. proper convex functions.

If the domain of one of these functions contains an interior point of the domain of the other, then

$$\inf_{x \in X} \{f(x) + g(x)\} = \max_{x^* \in X^*} \{-f^*(x^*) - g^*(-x^*)\}.$$

THEOREM (Simons, 1998).
Let $A : X \rightrightarrows X^*$ be monotone.

Then A is maximal monotone if and only if

$$Graph(A) + Graph(-J) = X \times X^*.$$

THEOREM (Torralba, 1996).
For every maximal monotone operator $A : X \rightrightarrows X^*$, if $\alpha, \beta > 0$ and $(x, x^*) \in X \times X^*$ are such that

$$\inf_{(y,y^*)\in Graph(A)} \langle x - y, x^* - y^* \rangle \geq -\alpha\beta$$

then there exists $(z, z^*) \in Graph\,(A)$ such that

$$\|z - x\| \leq \alpha \text{ and } \|z^* - x^*\| \leq \beta.$$

Proof.
$\dfrac{\alpha}{\beta}A$ is maximal monotone.
$\left(x, \dfrac{\alpha}{\beta}x^*\right) \in Graph\left(\dfrac{\alpha}{\beta}A\right) + Graph\,(-J)$
There exists $(z, z^*) \in Graph\,(A)$ such that

$$\left(x - z, \frac{\alpha}{\beta}x^* - \frac{\alpha}{\beta}z^*\right) \in Graph\,(-J)\,.$$

$$\|z - x\|^2 = \tfrac{\alpha^2}{\beta^2}\,\|z^* - x^*\|^2 = \tfrac{\alpha}{\beta}\,\langle z - x, x^* - z^* \rangle \leq \alpha^2$$
$$\|z^* - x^*\|^2 \leq \beta^2$$

COROLLARY. Let $A : X \rightrightarrows X^*$ be maximal monotone.

If φ_A is finite-valued
then $D(A)$ and $R(A)$ are dense in X and X^*, respectively.

THEOREM. For every monotone operator
$A : X \rightrightarrows X^*$, the following statements are equivalent:

(1) A is maximal monotone.

(2) $Graph(A) + Graph(-B) = X \times X^*$
for every maximal monotone operator $B : X \rightrightarrows X^*$ such that φ_B is finite-valued.

(3) There exist an operator
$B : X \rightrightarrows X^*$ such that

$$Graph(A) + Graph(-B) = X \times X^*$$

and $(p, p^*) \in Graph(B)$ such that

$$\langle p - y, p^* - y^* \rangle > 0 \quad \forall\ (y, y^*) \in Graph(B) \setminus \{(p, p^*)\}.$$

Proof of (1) $\Longrightarrow$ (2).
Let $\left(x_0, x_0^*\right) \in X \times X^*$.
Define $A' : X \rightrightarrows X^*$ by

$$Graph\left(A'\right) := Graph\left(A\right) - \left(x_0, x_0^*\right)$$

and $h : X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ by

$$h\left(x, x^*\right) := \varphi_B\left(-x, x^*\right).$$

$$\sigma_{A'}\left(x, x^*\right) + h\left(x, x^*\right) \geq \langle x, x^* \rangle + \langle -x, x^* \rangle = 0$$

There exists $\left(y, y^*\right) \in X \times X^*$ such that

$$\varphi_{A'}\left(y, y^*\right) + h^*\left(-y^*, -y\right) \leq 0$$

$$\begin{aligned} \varphi_{A'}\left(y, y^*\right) + h^*\left(-y^*, -y\right) &= \varphi_{A'}\left(y, y^*\right) + \sigma_B\left(-y, y^*\right) \\ &\geq \langle y, y^* \rangle + \langle -y, y^* \rangle = 0 \end{aligned}$$

$$\varphi_{A'}\left(y, y^*\right) = \langle y, y^* \rangle \text{ and } \sigma_B\left(-y, y^*\right) = \langle -y, y^* \rangle$$

$$\left(y, y^*\right) \in Graph\left(A'\right) \text{ and } \left(-y, y^*\right) \in Graph\left(B\right)$$

$$\begin{aligned} \left(x_0, x_0^*\right) &= \left(x_0, x_0^*\right) + \left(y, y^*\right) + \left(-y, -y^*\right) \\ &\in \left(x_0, x_0^*\right) + Graph\left(A'\right) + Graph\left(-B\right) \\ &= Graph\left(A\right) + Graph\left(-B\right) \end{aligned}$$

COROLLARY. For every maximal monotone operator $B : X \rightrightarrows X^*$,

φ_B is finite-valued
if and only if
$D(B) = X$, $R(B) = X^*$ and B satisfies the Brézis-Haraux condition.

Proof of "only if".
Take A with $D(A) = \{0\}$ and $R(A) = X^*$.
Take A with $D(A) = X$ and $R(A) = \{0\}$.

COROLLARY. Let $T : X \rightrightarrows X^*$ be maximal monotone.

If φ_T is finite-valued
then for every closed convex set $K \subseteq X$
the generalized variational inequality problem $GVI(T, K)$
has a solution, that is,
there exist $x \in K$ and $x^* \in T(x)$ such that

$$\langle y - x, x^* \rangle \geq 0 \qquad \forall\, y \in K.$$

Proof.

Take $A = N_K$ and

define $B : X \rightrightarrows X^*$ by $B(x) = -T(-x)$.

A and B are maximal monotone.

$\varphi_B(x, x^*) = \varphi_T(-x, -x^*) \qquad \forall\ (x, x^*) \in X \times X^*$.

$$(0, 0) \in Graph(N_K) + Graph(-B)$$

There exists $(x, y^*) \in Graph(N_K)$ such that

$(-x, -y^*) \in Graph(-B)$.

Take $x^* = -y^*$.

$(x, -x^*) = (x, y^*) \in Graph(N_K)$, that is,

$$\langle y - x, x^* \rangle \geq 0 \qquad \forall\ y \in K.$$

$x^* = -y^* \in -B(-x) = T(x)$

PROPOSITION.

Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a l.s.c. proper convex function. Then

$\varphi_{\partial f}$ is finite-valued $\Longleftrightarrow$ f and f^* are finite-valued

A l.s.c. proper convex function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is *supercoercive* if

$$\lim_{\|x\| \longrightarrow \infty} \frac{f(x)}{\|x\|} = +\infty.$$

f is *supercoercive* $\quad \Longrightarrow \quad f^*$ is finite-valued

COROLLARY.
Let $A : X \rightrightarrows X^*$ be a monotone operator and $f : X \longrightarrow \mathbb{R}$ be a l.s.c. proper convex function such that f^* is finite-valued.

Then A is maximal monotone if and only if

$$Graph\,(A) + Graph\,(-\partial f) = X \times X^*.$$

$A : X \rightrightarrows X^*$ is *strictly monotone* if

for $x, y \in X$ with $x \neq y$, $x^* \in A(x)$ and $y^* \in A(y)$,
$$\langle x - y, x^* - y^* \rangle > 0.$$

LEMMA.
If $A : X \rightrightarrows X^*$ is monotone and
$B : X \rightrightarrows X^*$ is strictly monotone
then $A + B$ is strictly monotone and
hence $(A + B)^{-1}$ is single-valued on its domain.

COROLLARY.
Let $A : X \rightrightarrows X^*$ be a monotone operator and
$B : X \rightrightarrows X^*$ be a maximal monotone operator with
finite-valued Fitzpatrick function φ_B.

If A is maximal monotone
then $R(A + B) = X^*$.

Conversely, if B is single-valued and strictly monotone
and $R(A + B) = X^*$
then A is maximal monotone.

THE NONREFLEXIVE CASE

X Banach space, $T : X \rightrightarrows X^*$

$\widetilde{T} : X^{**} \rightrightarrows X^*$
$Graph\left(\widetilde{T}\right) :=$
$\{(x^{**}, x^*) : \langle x^{**} - y, x^* - y^* \rangle \geq 0, \forall (y, y^*) \in \mathcal{G}(T)\}$

A monotone operator $T : X \rightrightarrows X^*$ is of type (D) if for every $(x^{**}, x^*) \in Graph\left(\widetilde{T}\right)$ there exists a bounded net (x_α, x_α^*) in $\mathcal{G}(T)$ such that $x_\alpha \to x^{**}$ in the $\sigma(X^{**}, X^*)$ topology of X^{**} and $x_\alpha^* \to x^*$ in the norm topology of X^*.

$\varrho : X \times X^* \to X \times X^*, (x, x^*) \mapsto (x, -x^*)$.

Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex functions. We call $z^* \in X^*$ a Fenchel functional for f and g if

$$f^*(z^*) + g^*(-z^*) \leq 0.$$

THEOREM.

Let $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type (D).

Then the following statements are equivalent:

(a) $R(\widetilde{S} + \widetilde{T}) = X^*$.

(b) for all $u^*, v^* \in X^*$, there exists $(x^{**}, x^*) \in X^{**} \times X^*$ such that, for all convex representations h of $u^* + S$ and k of $v^* + T$, (x^*, x^{**}) is a Fenchel functional for h and $k \circ \varrho$;

(c) for all $u^*, v^* \in X^*$, there exist convex representations h of $u^* + S$ and k of $v^* + T$ such that h and $k \circ \varrho$ have a Fenchel functional.

A sufficient condition for $\widetilde{S} + \widetilde{T}$ to be surjective is that, for all $w^* \in X^*$, there exist convex representations h of S and k of T such that

$$\bigcup_{\lambda > 0} \lambda[\mathrm{dom}\, h - \varrho(\mathrm{dom}\, k)] \text{ is a closed subspace of } X \times X^*.$$

POSITIVE SETS

S. Simons   Journal of Convex Analysis (2007)

$F \neq \{0\}$ real Banach space, F^* its dual

The monotone case: $F = X \times X^*$

$b : F \times F \longrightarrow \mathbb{R}$ continuous, symmetric, bilinear form that separates the points of F

The monotone case:

$b\left((x, x^*), (y, y^*)\right) = \langle x, y^* \rangle + \langle y, x^* \rangle$

$q : F \longrightarrow \mathbb{R}$ defined by $q(x) = \frac{1}{2} b(x, x)$

The monotone case: $q(x, x^*) = \langle x, x^* \rangle$

$A \subseteq F$ is q-*positive* if $a_1, a_2 \in A \Longrightarrow q(a_1 - a_2) \geq 0$

The monotone case:

A is q-positive $\Longleftrightarrow$ A is the graph of a monotone operator

$\Phi_{q,A} = \Phi_A : F \longrightarrow \mathbb{R} \cup \{+\infty\}$
$\Phi_A(x) = q(x) - \inf_{x \in A} q(x-a) = \sup_{x \in A} \{b(x,a) - q(a)\}$

The monotone case:

Φ_A is the Fitzpatrick function of the operator whose graph is A.

Φ_A is proper, convex and l.s.c..

$M \subseteq F$ is *maximally q-positive* if M is q–positive and not properly contained in any other q–positive set

The monotone case:

$$M \text{ is maximally } q\text{-positive}$$
$$\Longleftrightarrow$$
$$M \text{ is the graph of a maximal monotone operator}$$

If M is maximally q-positive then

$$\begin{array}{lllll} \Phi_M(x) & \geq & q(x) & \forall\, x \in F, & \\ \Phi_M(x) & = & q(x) & \Longleftrightarrow & x \in M. \end{array}$$

$h : F \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a *convex representation* of M if it is convex and l.s.c. and satisfies

$$\begin{array}{lllll} h(x) & \geq & q(x) & \forall\, x \in F, & \\ h(x) & = & q(x) & \Longleftrightarrow & x \in M. \end{array}$$

PROPOSITION.

Φ_M is the smallest convex representation of M.

Proof:

Let h be a convex representation of M, $x \in F$, $y \in M$ and $\lambda \in [0,1)$.

$$(1-\lambda)^2 q(x) + \lambda(1-\lambda) b(x,y) + \lambda^2 q(y)$$
$$= q((1-\lambda)x + \lambda y)$$
$$\leq h((1-\lambda)x + \lambda y)$$
$$\leq (1-\lambda) h(x) + \lambda h(y)$$
$$= (1-\lambda) h(x) + \lambda q(y)$$

$$(1-\lambda)^2 q(x) + \lambda(1-\lambda) b(x,y) - \lambda(1-\lambda) q(y)$$
$$\leq (1-\lambda) h(x)$$

$$(1-\lambda) q(x) + \lambda b(x,y) - \lambda q(y) \leq h(x)$$

$$b(x,y) - q(y) \leq h(x)$$

$$\Phi_M(x) \leq h(x)$$

PROPOSITION.

Let $A \subseteq F$. Then A is q-positive if and only if there exists a (l.s.c.) convex function $h : F \to \mathbb{R} \cup \{+\infty\}$ such that

$$\begin{array}{rclcr} h(x) & \geq & q(x) & \qquad \forall\, x \in F, \\ h(x) & = & q(x) & \qquad \Longleftarrow \qquad & x \in A. \end{array}$$

$i : F \longrightarrow F^*$

$i(y) = b(\cdot, y)$

The monotone case: $i(y, y^*) = \langle \cdot, y^* \rangle + \langle y, \cdot \rangle$

If X is reflexive then i is a surjective isometry.

$G = \left\{ x \in F : q(x) = -\frac{1}{2} \|x\|^2 \right\}$

The monotone case: $\qquad G = Graph(-J)$

THEOREM. Suppose that i is a surjective isometry and A is a nonempty q–positive subset of F. Then

$$A \text{ is maximally } q\text{-positive} \Longleftrightarrow A + G = F.$$

THEOREM. Suppose that i is a surjective isometry. Then for every q-positive set $A \subseteq F$, the following statements are equivalent:

(1) A is maximally q-positive.

(2) $A + C = F$ for every maximally $-q$-positive set $C \subseteq F$ such that $\Phi_{-q,C}$ is finite-valued.

(3) There exist a set $C \subseteq F$ such that $A + C = F$ and $p \in C$ such that

$$q\,(y - p) < 0 \qquad \forall\; y \in C \setminus \{p\}\,.$$