A Cyclic Algorithm for the Split Common Fixed Point Problem in Hilbert Spaces

Hong-Kun Xu

Department of Applied Mathematics

National Sun Yat-sen University

Kaohsiung, Taiwan

E-mail: xuhk@math.nsysu.edu.tw

Abstract

In this paper, a cyclic algorithm is constructed to approximate a solution to a common fixed point for two classes of cutter operators. As an application of this algorithm, we obtain a new method for solving the split common fixed point problems, as well as the multiple-sets split feasibility problems.

1 Introduction

The well-known convex feasibility problem (CFP) is to find a point x^* satisfying the property:

$$x^* \in \bigcap_{i=1}^{m} C_i \tag{1.1}$$

where $m \geq 1$ is an integer, and each C_i is a nonempty closed convex subset of a Hilbert space H.

CFP (1.1) is a special case of finding a common fixed point problem for nonlinear mappings:

$$x^* \in \bigcap_{i=1}^{m} \mathrm{Fix}(T_i) \tag{1.2}$$

where each $T_i : \mathcal{H} \to \mathcal{H}$ is a (nonlinear) mapping. If we take $T_i = P_{C_i}$, the metric projection from H onto C_i, then the common fixed point problem (1.2) is reduced to CFP (1.1).

It is an interesting problem to find out for what kind of mappings T_i one can solve (1.2) iteratively (assuming existence of solutions).

In the literature, there exists quite a lot of work for solving (1.2) for the class of nonexpansive mappings. Recall that a mapping $T : \mathcal{H} \to \mathcal{H}$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in \mathcal{H}.$$

A special case of CFP is the so-called split feasibility problem (SFP) [8]:

$$\text{finding } x^* \text{ such that } x^* \in C \text{ and } Ax^* \in Q \tag{1.3}$$

where C and Q are closed convex subsets of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, and $A : \mathcal{H} \to \mathcal{K}$ is a bounded linear operator.

Another special case of CFP is the so-called multiple set-split feasibility problem (MSSFP) [9]:

$$\text{finding } x^* \text{ such that } x^* \in \bigcap_{i=1}^{N} C_i \text{ and } Ax^* \in \bigcap_{j=1}^{M} Q_j \tag{1.4}$$

where C_i and Q_j are closed convex subsets of Hilbert spaces

$\mathcal{H}$ and $\mathcal{K}$, respectively, and $A : \mathcal{H} \to \mathcal{K}$ is a bounded linear operator. MSSFP (1.4) models intensity-modulated radiation therapy [11].

All of the problems above are special cases of the so-called split common fixed point problem (SCFP) which is formulated as a problem of finding a point x^* with the property:

$$x^* \in \bigcap_{i=1}^{p} \mathrm{Fix}(U_i) \quad \text{and} \quad Ax^* \in \bigcap_{i=1}^{r} \mathrm{Fix}(T_j), \tag{1.5}$$

where A is a bounded linear operator from a Hilbert space $\mathcal{H}$ to a second Hilbert space $\mathcal{K}$ and $U_i : \mathcal{H} \to \mathcal{H}, i = 1, \cdots, p$, $T_j : \mathcal{K} \to \mathcal{K}, j = 1, \cdots, r$, are nonlinear operators. In particular if $p = r = 1$, Problem (1.5) is reduced to find

$$x^* \in Fix(U), \quad Ax^* \in \mathrm{Fix}(T), \tag{1.6}$$

which is usually called the two sets-SCFP.

The concept of SCFP was first introduced by Censor and Segal [10] where they constructed an algorithm for the two sets-SCFP. Take an initial guess $x_1 \in \mathcal{H}$ and choose $0 < \gamma < 2/\|A\|^2$; and define a sequence (x_n) by the iterative procedure:

$$x_{n+1} = U(x_n - \gamma A^*(I - T)Ax_n). \tag{1.7}$$

According to this algorithm and using the product space technique, they introduced another algorithm for the general case, which generates a sequence (x_n) by the iterative procedure:

$$x_{n+1} = x_n + \gamma \left[\sum_{i=1}^{p} \alpha_i (U_i(x_n) - x_n) + \sum_{j=1}^{r} \beta_j A^*(T_j - I)Ax_n \right], \tag{1.8}$$

where $0 < \gamma < 2/L$ with $L = \sum_{i=1}^{p} \alpha_i + \|A\|^2 \sum_{j=1}^{r} \beta_j$. Under some suitable assumptions, the sequence generated by the above algorithm converges to a solution of SCFP (1.5).

It is obvious that Problem (1.6) is a particular case of the general SCFP (1.5). However the corresponding algorithm (1.8) for the general SCFP dose not reduce to algorithm (1.7) for Problem (1.6). It is the aim of this paper to introduce a new iterative algorithm for solving the general SCFP, including algorithm (1.7) as a special case. The paper is organized as follows. In the next section, some useful lemmas are given. In Section 3, we construct an algorithm for approximating a point $\hat{x}$ with the property:

$$\hat{x} \in \bigcap_{i=1}^{p} Fix(U_i) \quad \text{and} \quad \hat{x} \in \bigcap_{j=1}^{r} Fix(V_j), \tag{1.9}$$

where $V_i = I + (1/\|A\|^2)A^*(T_i - I)A$, whose solution set coincides with that of the SCFP. Finally this iterative method is applied to solve the general SCFP, as well as the MSSFP.

2 Preliminaries

Throughout this paper, we adopt the notation:

- I the identity operator on $\mathcal{H}$,
- $Fix(T)$ the set of the fixed points of an operator $T : \mathcal{H} \to \mathcal{H}$,
- $x_n \to x$ means (x_n) converges in norm to x,
- $x_n \rightharpoonup x$ means (x_n) converges in norm to x,
- $\omega_w(x_n)$ the set of the cluster points of (x_n) in the weak topology (i.e., the set $\{x : \exists x_{n_j} \rightharpoonup x\}$,
- Ω the solution set of SCFP (1.5).

Let $\mathcal{H}$ be a Hilbert space. Given $x, y \in \mathcal{H}$. We use $H(x, y)$ to denote the half-space determined by x, y; namely,

$$H(x, y) = \{u \in \mathcal{H} : \langle u - y, x - y\rangle \leq 0\}.$$

Definition 2.1. Assume that $T : \mathcal{H} \to \mathcal{H}$ is an operator with $Fix(T) \neq \emptyset$.

(a) We say that T is a cutter operator if $\mathrm{Fix}(T) \subset H(x, Tx)$ for $x \in \mathcal{H}$, or

$$\langle z - Tx, x - Tx \rangle \leq 0, \quad z \in \mathrm{Fix}(T), \quad x \in \mathcal{H}.$$

(b) We say that

- T is quasi-nonexpansive if

$$\|Tx - z\| \leq \|x - z\|, \quad z \in \mathrm{Fix}(T), \quad x \in \mathcal{H};$$

- T is strictly quasi-nonexpansive if

$$\|Tx - z\| < \|x - z\|, \quad z \in \mathrm{Fix}(T), \quad x \in \mathcal{H} \backslash \mathrm{Fix}(T);$$

- T is α-strongly quasi-nonexpansive, where $\alpha > 0$, if

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \alpha \|Tx - x\|^2, \quad z \in \mathrm{Fix}(T), \quad x \in \mathcal{H}.$$

(c) We say that $I - T$ is demiclosed at zero if

$$x_n \rightharpoonup x \quad \text{and} \quad (I - T)x_n \to 0 \quad \Rightarrow \quad (I - T)x = 0.$$

The concept of cutter operators was originally introduced by Bauschke and Combettes [2] and further studied by several authors (see for instance [6, 10]). Cutter operators are important because then include many types of nonlinear operators which arise in convex optimization.

Proposition 2.2. *Let $T : \mathcal{H} \to \mathcal{H}$ be a operator such that* $\mathrm{Fix}(T) \neq \emptyset$*. Then the following are equivalent:*

(i) T is a cutter;

(ii) there holds the relation:

$$\|x - Tx\|^2 \leq \langle x - z, x - Tx \rangle, \quad z \in \mathrm{Fix}(T), \quad x \in \mathcal{H};$$

(iii) there holds the relation:

$$\|z - Tx\|^2 \leq \|x - z\|^2 - \|x - Tx\|^2, \quad z \in \mathrm{Fix}(T), \quad x \in \mathcal{H};$$

consequently, a cutter is 1-strongly pseudo-nonexpansive.

The operator

$$T_\lambda := I + \lambda(T - I) = (1-\lambda)I + \lambda T, \quad \lambda \in (0,2)$$

is called a relaxation of T.

Lemma 2.3 ([6][2]). *(i) The fixed point set of a cutter operator T is closed convex; indeed,*

$$\mathrm{Fix}(T) = \bigcap_{x \in \mathcal{H}} H(x, Tx).$$

(ii) If T is a cutter, then so is the relaxation T_λ for $\lambda \in (0,1)$.

(iii) T is a cutter if and only if its relaxation T_λ is $(2-\lambda)/\lambda$-strongly quasi-nonexpansive:

$$\|T_\lambda x - z\|^2 \le \|x - z\|^2 - \frac{2-\lambda}{\lambda}\|T_\lambda x - x\|^2, \quad x \in \mathcal{H}, \; z \in \mathrm{Fix}(T). \tag{2.1}$$

In particular, T itself is 1-strongly quasi-nonexpansive.

Particularly, projections are cutter operators. Recall that, given a closed convex subset C of a Hilbert space $\mathcal{H}$, the projection $P_C : \mathcal{H} \to C$ assigns each $x \in \mathcal{H}$ to its closest point from C: defined by

$$P_C x = \arg\min_{z \in C} \|x - z\|.$$

It is well-known that $P_C x$ is characterized by the inequality:

$$P_C x \in C, \quad \langle x - P_C x, z - P_C x \rangle \leq 0, \quad z \in C. \qquad (2.2)$$

We end this section by introducing a concept, which plays an important role in convergence analysis for various iterative algorithms.

Definition 2.4. Assume that C is a closed convex nonempty subset and (x_n) is a sequence in $\mathcal{H}$. The sequence (x_n) is called Fejér monotone with respect to C, if

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \quad n \geq 1, \ z \in C.$$

Lemma 2.5. *If a sequence (x_n) is Fejér monotone with respect to a closed convex subset C, then the following hold.*

(a) $x_n \rightharpoonup x^* \in C$ *if and only if* $\omega_w(x_n) \subseteq C$;

(b) the sequence $\{P_C x_n\}$ *converges strongly to some point in* C*;*

(c) if $x_n \rightharpoonup x^* \in C$, *then* $x^* = \lim_{n\to\infty} P_C x_n$.

Proof. (a) and (b) are taken from [1, Theorem 2.16]. To show (c), let $\hat{x}$ be the limit of the sequence $\{P_C x_n\}$. It follows from the characterizing inequality (2.2) that

$$\langle x_n - P_C x_n, x^* - P_C x_n \rangle \leq 0.$$

Letting $n \to \infty$ yields

$$\langle x^* - \hat{x}, x^* - \hat{x} \rangle \leq 0,$$

that is, $x^* = \hat{x}$ and thus the proof is complete. □

3 Main results

First let us show the solution set of SCFP (1.5) coincides with that of common fixed point problem (1.9). To see this, we need the following lemma.

Lemma 3.1. *Let $A : \mathcal{H} \to \mathcal{K}$ be a given bounded linear operator and $T : \mathcal{K} \to \mathcal{K}$ be a cutter operator on $\mathcal{K}$. Assume that the equation*

$$(I - T)Ax = 0 \tag{3.1}$$

has a nonempty solution set. Then, for each constant $0 < \sigma \leq 1/\|A\|^2$, the operator

$$V := I + \sigma A^*(T - I)A \tag{3.2}$$

is a cutter on $\mathcal{H}$; moreover,

$$\mathrm{Fix}(V) = \{x \in \mathcal{H} : Ax \in \mathrm{Fix}(T)\} = A^{-1}(\mathrm{Fix}(T)). \tag{3.3}$$

Proof. We first verify (3.3). Observe that the assumption of existence of solutions of equation (3.1) implies that $\mathrm{Fix}(V)$ is nonempty. Observe also that $z \in \mathrm{Fix}(V)$ if and only if $A^*(T-I)Az = 0$. Since the inclusion $A^{-1}(Fix(T)) \subseteq Fix(V)$ is evident, we only need to show the converse inclusion $A^{-1}(Fix(T)) \supseteq Fix(V)$. To see this, we take an arbitrary $z \in Fix(V)$; then $A^*(T-I)Az = 0$. Take also an element $x \in A^{-1}(\mathrm{Fix}(T))$ so that $\langle (T-I)Az, T(Az)-Ax\rangle \le 0$ for $Ax \in \mathrm{Fix}(T)$ and T is a cutter. It turns out that

$$\begin{aligned}\|(T-I)Az\|^2 &= \langle (T-I)Az, T(Az)-Ax\rangle + \langle (T-I)Az, Ax-Az\rangle \\ &\le \langle (T-I)Az, Ax-Az\rangle \\ &= \langle A^*(T-I)Az, x-z\rangle \\ &= 0.\end{aligned}$$

It turns out that $Az = T(Az)$; hence $z \in A^{-1}(Fix(T))$.

We next turn to prove that V is a cutter operator on $\mathcal{H}$.

Let $z \in \mathrm{Fix}(V)$. It then follows that

$$\begin{aligned}\frac{1}{\sigma}\langle z - Vx, x - Vx\rangle &= \langle z - x - \sigma A^*(T-I)Ax, A^*(I-T)Ax\rangle \\ &= \langle z - x, A^*(I-T)Ax\rangle + \sigma\|A^*(I-T)Ax\|^2 \\ &= \langle Az - Ax, (I-T)Ax\rangle + \sigma\|A^*(I-T)Ax\|^2 \\ &= \langle Az - T(Ax), (I-T)Ax\rangle \\ &\quad + \sigma\|A^*(I-T)Ax\|^2 - \|(I-T)Ax\|^2. \end{aligned} \tag{3.4}$$

Since T is a cutter and $Az \in \mathrm{Fix}(T)$, $\langle Az - T(Ax), (I - T)Ax\rangle \leq 0$. Since also $\sigma\|A^*(I-T)Ax\|^2 \leq \sigma\|A\|^2\|(I - T)Ax\|^2 \leq \|(I-T)Ax\|^2$ for $0 < \sigma \leq 1/\|A\|^2$, we therefore immediately get from (3.4) that $\langle z - Vx, x - Vx\rangle \leq 0$ and V is a cutter. □

It is readily seen that the solution set of SCFP (1.5) and Problem (1.9) are identical by the preceding lemma. So solving SCFP (1.5) is equivalent to finding a common fixed point for two classes of cutter operators. Using this idea, we now introduce an algorithm to solve SCFP (1.5).

Take an initial guess $x_1 \in \mathcal{H}$ and $\lambda \in (0, 2)$ and define a sequence (x_n) by the following iterative procedure:

$$x_{n+1} = U_{[n]}[x_n + \lambda(V_{[n]}x_n - x_n)], \tag{3.5}$$

where $[n] := n \mod p$ with the mod function taking values in $\{1, \cdots, p\}$.

Theorem 3.2. *Let U_i and V_i be cutter operators on a Hilbert space $\mathcal{H}$ for $i = 1, 2, \cdots, p$. Suppose that $U_i - I$ and $V_i - I$ are demiclosed at zero for every $i = 1, 2, \cdots, p$. Assume that the solution set Ω of Problem* (1.9) *(with $r = p$) is nonempty. Then the sequence (x_n) generated by the algorithm* (3.5) *converges weakly to a point $x^* \in \Omega$. Moreover $x^* = \lim_{n\to\infty} P_\Omega x_n$.*

Proof. Let $z \in \Omega$ and set $V_{\lambda,n} = I + \lambda(V_{[n]} - I)$. Since $V_{\lambda,n}$ is $(2 - \lambda)/\lambda$-strongly quasi-nonexpansive, we deduce from (2.1) that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|U_{[n]}V_{\lambda,n}x_n - z\|^2 \\ &\le \|V_{\lambda,n}x_n - z\|^2 \\ &\le \|x_n - z\|^2 - \frac{2-\lambda}{\lambda}\|V_{\lambda,n}x_n - x_n\|^2 \\ &= \|x_n - z\|^2 - \lambda(2-\lambda)\|V_{[n]}x_n - x_n\|^2. \end{aligned} \tag{3.6}$$

Thus (x_n) is Fejér monotone with respect to Ω and

$$\sum_{n\ge1} \|V_{[n]}x_n - x_n\|^2 < \infty$$

and therefore

$$\|V_{[n]}x_n - x_n\| \to 0.$$

Let $x^* \in \omega_w(x_n)$ and let an index i be fixed. Take a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \rightharpoonup x^*$. It follows again from (2.1) that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|U_{[n]}(V_{\lambda,n}x_n) - V_{\lambda,n}x_n + V_{\lambda,n}x_n - x_n\|^2 \\
&\leq 2(\|U_{[n]}(V_{\lambda,n}x_n) - V_{\lambda,n}x_n\|^2 + \|V_{\lambda,n}x_n - x_n\|^2) \\
&\leq 2(\|V_{\lambda,n}x_n - z\|^2 - \|U_{[n]}V_{\lambda,n}x_n - z\|^2) \\
&\quad + \frac{2(2-\lambda)}{\lambda}(\|x_n - z\|^2 - \|V_{\lambda,n}x_n - z\|^2) \\
&\leq \nu(\|x_n - z\|^2 - \|x_{n+1} - z\|^2),
\end{aligned}$$

where $\nu = 2\max(1, (2-\lambda)/\lambda)$. Thus

$$\sum_{n\geq 1} \|x_{n+1} - x_n\|^2 < \infty.$$

This implies (see the proof of [2, Theorem 5.3]) that there exists a strictly increasing sequence (m_k) in $\mathbb{N}$ such that

$x_{m_k} \rightharpoonup x^*$ and $[m_k] = i$ for all k. It turns out that

$$\|V_i x_{m_k} - x_{m_k}\| = \|V_{m_k} x_{m_k} - x_{m_k}\| \to 0.$$

By the demiclosedness of $V_i - I$ at zero, $x^* \in Fix(V_i)$.

Set $y_{m_k} = x_{m_k} + \lambda(V_i x_{m_k} - x_{m_k})$; then $y_{m_k} \rightharpoonup x^*$ as $\|V_i x_{m_k} - x_{m_k}\| \to 0$.

It follows from (3.6) that

$$\begin{aligned}\|x_{m_k+1} - z\|^2 &= \|U_i y_{m_k} - z\| \\ &\leq \|y_{m_k} - z\|^2 \\ &\leq \|x_{m_k} - z\|^2 - \lambda(2-\lambda)\|V_i x_{m_k} - x_{m_k}\|^2.\end{aligned}$$

Hence $\lim_{k\to\infty} \|x_{m_k} - z\|$ coincides with $\lim_{k\to\infty} \|y_{m_k} - z\|$. Moreover

$$\begin{aligned}\|U_i y_{m_k} - y_{m_k}\|^2 &\leq \|y_{m_k} - z\|^2 - \|U_i y_{m_k} - z\|^2 \\ &= \|y_{m_k} - z\|^2 - \|x_{m_k+1} - z\|^2 \\ &\to 0 \quad (as\ k \to \infty).\end{aligned}$$

Since $U_i - I$ is demiclosedness at zero, $x^* \in Fix(U_i)$. Since this is true for every i, we get that $\omega_w(x_n) \subset \Omega$. By Lemma 2.5, we conclude that the sequence (x_n) converges weakly to a point in Ω. $\square$

We now consider SCFP under the assumption "$p = r$" and we will show this assumption is not restrictive.

We next consider SCTP (1.5) with $r = p$. Take an initial guess $x_1 \in \mathcal{H}$ and $0 < \gamma < 2/\|A\|^2$, and define a sequence (x_n) by the iterative procedure:

$$x_{n+1} = U_{[n]}[x_n + \gamma A^*(T_{[n]} - I)Ax_n], \tag{3.7}$$

where $[n] := n \mod p$ and the mod function takes values in $\{1, 2, \cdots, p\}$.

Theorem 3.3. *Let $p \geq 1$ be an integer, and for each $1 \leq i \leq p$, let U_i and T_i be cutter operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Suppose that SCTP* (1.5) *with $r = p$ has a nonempty solution set Ω. Suppose also that, for each $1 \leq i \leq p$, $U_i - I$ and $T_i - I$ are demiclosed at zero. Then the sequence (x_n) generated by the algorithm* (3.7) *converges weakly to a point in Ω. Moreover $x^* = \lim_{n\to\infty} P_\Omega x_n$.*

Proof. Take $0 < \sigma \leq 1/\|A\|^2$ such that $\gamma/\sigma < 2$ (e.g., $\sigma = 1/\|A\|^2$) and set

$$V_i = I + \sigma A^*(T_i - I)A, \quad i = 1, 2, \cdots, p.$$

By Lemma 3.1, V_i is a cutter. Let

$$U_{[n]} := U_{n \mod p} \quad \text{and} \quad V_{[n]} := V_{n \mod p}.$$

We can rewrite (3.7) as

$$x_{n+1} = U_{[n]}[x_n + \lambda(V_{[n]}x_n - x_n)], \tag{3.8}$$

where $\lambda = \gamma/\sigma \in (0, 2)$.

We next prove the demiclosedness of the operator $V_i - I$ at zero for every $i = 1, 2, \cdots, p$. To see this, assume $z_n \rightharpoonup z$ and $z_n - V_i z_n \to 0$ as $n \to \infty$. From the definition of V_i it follows that

$$\|A^*(I - T_i)Az_n\| = \frac{1}{\sigma}\|z_n - V_i z_n\| \to 0. \tag{3.9}$$

Take $q \in \Omega$. Since T_i is a cutter, we arrive at

$$\begin{aligned}
\|Av_n - T_i(Av_n)\|^2 &= \langle (I - T_i)Av_n, Av_n - Aq\rangle \\
&\quad + \langle (I - T_i)Av_n, Az - T_i(Av_n)\rangle \\
&\leq \langle Av_n - T_i(Av_n), Av_n - Aq\rangle \\
&= \langle A^*(I - T_i)Az_n, x_n - q\rangle \\
&\leq M\|A^*(I - T_i)Az_n\|,
\end{aligned}$$

where M is a constant such that $M \geq \|v_n - q\|$ for all n. It turns from (3.9) that

$$\|Az_n - T_i(Az_n)\| \to 0, \quad n \to \infty.$$

However, the weak continuity of A yields that $Az_n \rightharpoonup Az$, which together with the demiclosedness of $I - T_i$ at zero enables us to deduce

$$Az = T_i(Az) \Rightarrow z \in Fix(V_i).$$

This shows that $V_i - I$ is demiclosed at zero for every $i = 1, 2, \cdots, p$. Applying Lemma 3.1 and Theorem 3.2 immediately gets the result as desired. □

Remark 3.4. The assumption "$p = r$" above is not restrictive. In fact, if $p < r$, we define in (3.7) $[n] = n \mod r, U_i = I$ for $p+1 \leq i \leq r$; otherwise, define $T_i = I$ for $r+1 \leq i \leq p$.

Remark 3.5. For the particular case "$p = r = 1$," our algorithm reduces to Censor and Segal's algorithm (1.7) for solving the two-sets split common fixed point problems.

By using the preceding results, we immediately get a new algorithm for solving MSSFP. Also assume that $p = r$ without loss of generality. Take an initial guess $x_1 \in \mathcal{H}$ and choose $\gamma \in (0, 2/\|A\|^2)$; and define a sequence (x_n) by the iterative procedure:

$$x_{n+1} = P_{C_{[n]}}[x_n + \gamma A^*(P_{Q_{[n]}} - I)Ax_n], \qquad (3.10)$$

where $[n] = n \mod p$ and the mod m function takes values in $\{1, 2, \cdots, p\}$.

Corollary 3.6. *The sequence* (x_n)*, generated by* (3.10)*, converges weakly to a solution of MSSFP whenever its solution set is nonempty.*

Remark 3.7. For the particular case "$p = r = 1$," our algorithm reduces to Byrne's CQ algorithm (see [3, 4]) for solving the split feasibility problems.

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