Splitting methods for constructing the resolvent of a sum of maximal monotone operators

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Overview: Solving strongly monotone inclusions

Throughout, $\mathcal{H}$ is a real Hilbert space.

- Given $r \in \mathcal{H}$ and maximal monotone operators $(B_i)_{1\leq i\leq m}$ acting on $\mathcal{H}$, with B_1 strongly monotone,

$$\text{Find} \quad x \in \mathcal{H} \quad \text{such that} \quad r \in \sum_{i=1}^{m} B_i x.$$

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- *Equivalent formulation:* Given $r \in \mathcal{H}$ and maximal monotone operators $(A_i)_{1 \le i \le m}$ on $\mathcal{H}$, weights $(\omega_i)_{1 \le i \le m}$ in $]0,1[$ such that $\sum_{i=1}^{m} \omega_i = 1$, solve $r \in \sum_{i=1}^{m} \omega_i A_i x$, i.e., compute

$$x = (\mathrm{Id} + A)^{-1} r = J_A r, \quad \text{where} \quad A = \sum_{i=1}^{m} \omega_i A_i.$$

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- We propose two algorithms to construct $J_A r$.

PART I: Douglas-Rachford splitting

A first algorithm to construct $J_A r$.

Douglas-Rachford splitting for two monotone operators

Douglas-Rachford (1956), Lieutaud (1969), Lions-Mercier (1979), Eckstein-Bertsekas (1992),...

Algorithm 1

$(\mathcal{H}, |||\cdot|||)$ a real Hilbert space, $\boldsymbol{A}$ and $\boldsymbol{B}$ maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ such that $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B}) \neq \varnothing$, $\gamma \in \,]0,+\infty[$, $(\lambda_n)_{n\in\mathbb{N}}$ in $]0,2]$, and $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$.
Routine:

Initialization

$$\left\lfloor \boldsymbol{z}_0 \in \mathcal{H} \right.$$

For $n = 0, 1, \ldots$

$$\left\lfloor \begin{array}{ll} \boldsymbol{y}_n & = J_{\gamma\boldsymbol{B}}\boldsymbol{z}_n + \boldsymbol{b}_n \\ \boldsymbol{z}_{n+1} & = \boldsymbol{z}_n + \lambda_n\big(J_{\gamma\boldsymbol{A}}(2\boldsymbol{y}_n - \boldsymbol{z}_n) + \boldsymbol{a}_n - \boldsymbol{y}_n\big). \end{array} \right.$$

Douglas-Rachford splitting for two monotone operators

Theorem

- *Suppose that* $\sum_{n\in\mathbb{N}} \lambda_n(|||\boldsymbol{a}_n||| + |||\boldsymbol{b}_n|||) < +\infty$, $\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$, *and* $(\forall n \in \mathbb{N})\ \lambda_n < 2$. *Then:*
  - $(\boldsymbol{z}_n)_{n\in\mathbb{N}}$ *converges weakly to a point* $\boldsymbol{z} \in \mathcal{H}$ *and* $J_{\gamma\boldsymbol{B}}\boldsymbol{z}$ *is a zero of* $\boldsymbol{A}+\boldsymbol{B}$ *[PLC, 2004].*

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  - *Suppose that* $\boldsymbol{A} = N_{\boldsymbol{D}}$, *where* $\boldsymbol{D}$ *is a closed affine subspace of* $\mathcal{H}$. *Then* $J_{\gamma\boldsymbol{A}}\boldsymbol{z}_n \rightharpoonup \boldsymbol{y} \in \operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$.
  - *Suppose that* $\boldsymbol{A} = N_{\boldsymbol{D}}$, *where* $\boldsymbol{D}$ *is a closed vector subspace of* $\mathcal{H}$, *and that* $\boldsymbol{b}_n \rightharpoonup 0$. *Then* $J_{\gamma\boldsymbol{A}}\boldsymbol{y}_n \rightharpoonup \boldsymbol{y} \in \operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$.
- *Suppose that* $\sum_{n\in\mathbb{N}} |||\boldsymbol{a}_n||| < +\infty$, $\sum_{n\in\mathbb{N}} |||\boldsymbol{b}_n||| < +\infty$, $\inf_{n\in\mathbb{N}} \lambda_n > 0$, *and* $\boldsymbol{B}$ *is uniformly monotone on the bounded subsets of* $\mathcal{H}$. *Then* $\boldsymbol{y}_n \to \boldsymbol{y} \in \operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$.

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Douglas-Rachford splitting for two monotone operators

Setting $A = N_{\boldsymbol{D}}$, where $\boldsymbol{D}$ is a closed affine subspace of $\mathcal{H}$, we obtain:

Algorithm 2

$$\boldsymbol{z}_0 \in \mathcal{H} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{ll} \boldsymbol{y}_n & = J_{\gamma \boldsymbol{B}} \boldsymbol{z}_n + \boldsymbol{b}_n \\ \boldsymbol{x}_n & = P_{\boldsymbol{D}} \boldsymbol{y}_n \\ \boldsymbol{p}_n & = P_{\boldsymbol{D}} \boldsymbol{z}_n \\ \boldsymbol{z}_{n+1} & = \boldsymbol{z}_n + \lambda_n (2\boldsymbol{x}_n - \boldsymbol{p}_n - \boldsymbol{y}_n). \end{array} \right.$$

Corollary

- *Suppose that* $\sum_{n \in \mathbb{N}} \lambda_n |||\boldsymbol{b}_n||| < +\infty$, $\sum_{n \in \mathbb{N}} \lambda_n (2 - \lambda_n) = +\infty$, *and* $(\forall n \in \mathbb{N})$ $\lambda_n < 2$. *Then* $\boldsymbol{p}_n \rightharpoonup \boldsymbol{p} \in \operatorname{zer}(N_{\boldsymbol{D}} + \boldsymbol{B})$.
- *Suppose that* $\sum_{n \in \mathbb{N}} |||\boldsymbol{b}_n||| < +\infty$, $\inf_{n \in \mathbb{N}} \lambda_n > 0$, *and* $\boldsymbol{B}$ *is uniformly monotone on the bounded subsets of* $\mathcal{H}$. *Then* $\boldsymbol{x}_n \to \boldsymbol{x} \in \operatorname{zer}(N_{\boldsymbol{D}} + \boldsymbol{B})$.

Splitting for m monotone operators

- $(B_i)_{1\leq i\leq m}$ are maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$, and

$$B=\sum_{i=1}^{m}\omega_i B_i,\quad \text{where}\quad \{\omega_i\}_{1\leq i\leq m}\subset\,]0,1[\quad \text{and}\quad \sum_{i=1}^{m}\omega_i=1.$$

- $\boldsymbol{\mathcal{H}}$ is the m-fold Cartesian product of $\mathcal{H}$ with scalar product $(\boldsymbol{x},\boldsymbol{y})\mapsto\sum_{i=1}^{m}\omega_i\langle x_i \mid y_i\rangle$.
- $\boldsymbol{A}=N_{\boldsymbol{D}}$, where $\boldsymbol{D}=\big\{(x,\ldots,x)\in\boldsymbol{\mathcal{H}}\mid x\in\mathcal{H}\big\}$.
- $\boldsymbol{B}\colon\boldsymbol{\mathcal{H}}\to 2^{\boldsymbol{\mathcal{H}}}\colon\boldsymbol{x}\mapsto \bigtimes_{i=1}^{m} B_i x_i$.
- $\boldsymbol{j}\colon\mathcal{H}\to\boldsymbol{D}\colon x\mapsto(x,\ldots,x)$.
- Thus, $\boldsymbol{j}(\operatorname{zer}B)=\operatorname{zer}(N_{\boldsymbol{D}}+\boldsymbol{B})$.

Splitting for m monotone operators

Algorithm 3

Initialization

For $i = 1, \ldots, m$

$z_{i,0} \in \mathcal{H}$

For $n = 0, 1, \ldots$

For $i = 1, \ldots, m$

$y_{i,n} = J_{\gamma B_i} z_{i,n} + b_{i,n}$

$x_n = \sum_{i=1}^m \omega_i y_{i,n}$

$p_n = \sum_{i=1}^m \omega_i z_{i,n}$

$\lambda_n \in]0, 2]$

For $i = 1, \ldots, m$

$z_{i,n+1} = z_{i,n} + \lambda_n \big(2x_n - p_n - y_{i,n}\big).$

Splitting for m monotone operators

Proposition

- *Suppose that* $\max_{1\leq i\leq m}\sum_{n\in\mathbb{N}}\lambda_n\|b_{i,n}\|<+\infty$, $\sum_{n\in\mathbb{N}}\lambda_n(2-\lambda_n)=+\infty$, *and* $(\forall n\in\mathbb{N})\ \lambda_n<2$. *Then:*
  - $p_n\rightharpoonup p\in\operatorname{zer}B$.
  - *Suppose that* $(\forall i\in\{1,,\ldots,m\})\ b_{i,n}\rightharpoonup 0$. *Then* $x_n\rightharpoonup x\in\operatorname{zer}B$.
- *Suppose that* $\max_{1\leq i\leq m}\sum_{n\in\mathbb{N}}\|b_{i,n}\|<+\infty$, $\inf_{n\in\mathbb{N}}\lambda_n>0$, *and the* B_i*s are strongly monotone. Then* $x_n\to x\in\operatorname{zer}B$. ▲▲

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Remark

A special case of ▲▲ was obtained by Spingarn (1983) via the method of partial inverses.

Splitting for m monotone operators

Spingarn's splitting algorithm

Initialization

$s_0 \in \mathcal{H}$

$(v_{i,0})_{1 \le i \le m} \in \mathcal{H}^m$ satisfy $\sum_{i=1}^m \omega_i v_{i,0} = 0$

For $n = 0, 1, \ldots$

For $i = 1, \ldots, m$

find $(y_{i,n}, u_{i,n}) \in \operatorname{gr} B_i$ such that $y_{i,n} + u_{i,n} = s_n + v_{i,n}$

$$s_{n+1} = \sum_{i=1}^m \omega_i y_{i,n}$$

$$q_n = \sum_{i=1}^m \omega_i u_{i,n}$$

For $i = 1, \ldots, m$

$v_{i,n+1} = u_{i,n} - q_n.$

Splitting for the resolvent of the sum

Back to our problem...

- $(A_i)_{1\le i\le m}$ are maximal monotone
- Set

$$A=\sum_{i=1}^{m}\omega_i A_i,\quad \text{where}\quad \{\omega_i\}_{1\le i\le m}\subset\,]0,1[\quad\text{and}\quad \sum_{i=1}^{m}\omega_i=1.$$

- Let $r\in\operatorname{ran}(\mathrm{Id}+A)$
- The goal is to construct $J_A\,r$.

Splitting for the resolvent of the sum

Algorithm 4

Let $\gamma \in \,]0,+\infty[$, $(\lambda_n)_{n\in\mathbb{N}}$ in $]0,2]$, and, for every $i \in \{1,\ldots,m\}$, $(a_{i,n})_{n\in\mathbb{N}}$ in $\mathcal{H}$.

Initialization
- For $i = 1,\ldots,m$
  - $z_{i,0} \in \mathcal{H}$

For $n = 0,1,\ldots$
- For $i = 1,\ldots,m$
  - $y_{i,n} = J_{\frac{\gamma}{\gamma+1}A_i}\Big(\big(z_{i,n}+\gamma r\big)/(\gamma+1)\Big) + a_{i,n}$
- $x_n = \sum_{i=1}^m \omega_i y_{i,n}$
- $p_n = \sum_{i=1}^m \omega_i z_{i,n}$
- For $i = 1,\ldots,m$
  - $z_{i,n+1} = z_{i,n} + \lambda_n\big(2x_n - p_n - y_{i,n}\big).$

Splitting for the resolvent of the sum

Proposition

Suppose that $\inf_{n\in\mathbb{N}} \lambda_n > 0$ *and that* $\max_{1\leq i\leq m} \sum_{n\in\mathbb{N}} \|a_{i,n}\| < +\infty$. *Then* $x_n \to J_A\, r$.

Proof: Set

$$(\forall i \in \{1,\ldots,m\}) \quad B_i \colon \mathcal{H} \to 2^{\mathcal{H}} \colon y \mapsto -r + y + A_i y$$

in ▲▲.

The proximity operator of the sum

- Let $(f_i)_{1\leq i\leq m}$ be functions in $\Gamma_0(\mathcal{H})$ such that $\bigcap_{i=1}^m \operatorname{dom} f_i \neq \varnothing$.

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- For every $r \in \mathcal{H}$,

$$\operatorname{prox}_f r = \operatorname{argmin}_{x\in\mathcal{H}} f(x) + \frac{1}{2}\|r - x\|^2$$

is uniquely defined.

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  is uniquely defined.
- Setting $A_i = \partial f_i$ in Algorithm 4 and assuming some CQ so that $\partial f = \sum_{i=1}^m \omega_i A_i$, we can construct $\operatorname{prox}_f r$.

Some applications of splitting in signal processing

- PLC and V. R. Wajs, “Signal recovery by proximal forward-backward splitting,” *Multiscale Model. Simul.,* vol. 4, pp. 1168-1200, 2005.
- C. Chaux, PLC, J.-C. Pesquet, and V. R. Wajs, “A variational formulation for frame-based inverse problems,” *Inverse Problems,* vol. 23, pp. 1495-1518, 2007.
- PLC and J.-C. Pesquet, “Proximal thresholding algorithm for minimization over orthonormal bases,” *SIAM J. Optim.,* vol. 18, pp. 1351–1376, 2007.
- PLC and J.-C. Pesquet, “A Douglas-Rachford splitting approach to nonsmooth convex variational signal recovery,” *IEEE J. Selected Topics Signal Process.,* vol. 1, pp 564–574, 2007.
- PLC and J.-C. Pesquet, “A proximal decomposition method for solving convex variational inverse problems,” vol. 24, 2008.

PART II: A Dykstra-like approach

A second algorithm to construct $J_A r$.

Von Neumann's alternating projections algorithm

Theorem (von Neumann, 1933)

Let $r \in \mathcal{H}$, let U and V be closed vector subspaces of $\mathcal{H}$, and set

$$y_0 = r \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{ll} x_n & = P_V y_n \\ y_{n+1} & = P_U x_n. \end{array} \right.$$

Then $x_n \to P_{U \cap V} r$.

- Von Neumann's theorem is a best approximation result.
- If U and V are intersecting closed convex subsets of $\mathcal{H}$, we merely have $x_n \rightharpoonup x$, where $x \in U \cap V$ is undetermined (Bregman, 1965).

Dykstra's alternating projections algorithm

Theorem (Boyle/Dykstra, 1986)

Let $z \in \mathcal{H}$, let C and D be closed convex subsets *of $\mathcal{H}$ such that $C \cap D \neq \varnothing$, and set*

$$\left\lfloor \begin{array}{l} y_0 = r \\ \\ \end{array} \right. \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{ll} x_n & = P_D(y_n \qquad) \\ y_{n+1} & = P_C(x_n \qquad) \\ & \qquad\qquad . \end{array} \right.$$

Then $x_n \rightharpoonup x \in C \cap D$ [Bregman (1965)] .

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Let $z \in \mathcal{H}$, let C and D be closed convex subsets of $\mathcal{H}$ such that $C \cap D \neq \varnothing$, and set

$$\left\lfloor \begin{array}{l} y_0 = r \\ p_0 = 0 \\ q_0 = 0 \end{array} \right. \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{ll} x_n & = P_D(y_n + q_n) \\ q_{n+1} & = y_n + q_n - x_n \\ y_{n+1} & = P_C(x_n + p_n) \\ p_{n+1} & = x_n + p_n - y_{n+1}. \end{array} \right.$$

Then $x_n \to P_{C \cap D} r$.

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Then $x_n \longrightarrow P_{C\cap D} r$.

- Von Neumann's theorem is a special case.
- Nontrival incremental proofs: Dykstra 1983, Han 1988, Boyle/Dykstra 1986, Gaffke/Mathar 1989, De Pierro/Iusem 1991, Bauschke/Borwein 1994, etc.

The resolvent of the sum of two monotone operators

Theorem (Bauschke/PLC, 2008)

Let $(\mathcal{H}, |||\cdot|||)$ *be a real Hilbert space, and let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be maximal monotone operators from* $\mathcal{H}$ *to* $2^{\mathcal{H}}$.

Furthermore, let $\boldsymbol{r} \in \operatorname{ran}(\mathbf{Id} + \boldsymbol{A} + \boldsymbol{B})$ *and let* $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ *be the sequence generated by the following routine.*

$$\left\lfloor \begin{array}{l} \boldsymbol{y}_0 = \boldsymbol{r} \\ \boldsymbol{p}_0 = 0 \\ \boldsymbol{q}_0 = 0 \end{array} \right. \quad \text{and} \quad \left\lfloor \begin{array}{ll} \boldsymbol{x}_n & = J_{\boldsymbol{B}}(\boldsymbol{y}_n + \boldsymbol{q}_n) \\ \boldsymbol{q}_{n+1} & = \boldsymbol{y}_n + \boldsymbol{q}_n - \boldsymbol{x}_n \\ \boldsymbol{y}_{n+1} & = J_{\boldsymbol{A}}(\boldsymbol{x}_n + \boldsymbol{p}_n) \\ \boldsymbol{p}_{n+1} & = \boldsymbol{x}_n + \boldsymbol{p}_n - \boldsymbol{y}_{n+1}. \end{array} \right.$$

Then $\boldsymbol{x}_n \to J_{\boldsymbol{A}+\boldsymbol{B}}\, \boldsymbol{r}$.

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Let $(\mathcal{H}, |||\cdot|||)$ *be a real Hilbert space, and let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be maximal monotone operators from* $\mathcal{H}$ *to* $2^{\mathcal{H}}$. *Let* $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$ *and* $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$ *be sequences in* $\mathcal{H}$ *such that*

$$\sum_{n\in\mathbb{N}} |||\boldsymbol{a}_n||| < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} |||\boldsymbol{b}_n||| < +\infty.$$

Furthermore, let $\boldsymbol{r} \in \operatorname{ran}(\mathbf{Id} + \boldsymbol{A} + \boldsymbol{B})$ *and let* $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ *be the sequence generated by the following routine.*

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Then $\boldsymbol{x}_n \to J_{\boldsymbol{A}+\boldsymbol{B}}\, \boldsymbol{r}$.

Proof outline

- Note that $\boldsymbol{q}_{n+1} + \boldsymbol{p}_n + \boldsymbol{x}_n = \boldsymbol{y}_n + \boldsymbol{q}_n + \boldsymbol{p}_n$ and $\boldsymbol{q}_n + \boldsymbol{p}_n = \boldsymbol{r} - \boldsymbol{y}_n$.
- Hence $\boldsymbol{r} = \boldsymbol{y}_n + \boldsymbol{q}_n + \boldsymbol{p}_n = \boldsymbol{q}_{n+1} + \boldsymbol{p}_n + \boldsymbol{x}_n$.
- Rewrite algorithm as

$$\left\lfloor \begin{array}{l} \boldsymbol{y}_0 = \boldsymbol{r} \\ \boldsymbol{q}_0 = 0 \\ \boldsymbol{p}_0 = 0 \end{array} \right. \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{ll} \boldsymbol{x}_n & = J_{\boldsymbol{B}}(\boldsymbol{r} - \boldsymbol{p}_n) + \boldsymbol{b}_n \\ \boldsymbol{q}_{n+1} & = \boldsymbol{r} - \boldsymbol{p}_n - \boldsymbol{x}_n \\ \boldsymbol{y}_{n+1} & = J_{\boldsymbol{A}}(\boldsymbol{r} - \boldsymbol{q}_{n+1}) + \boldsymbol{a}_n \\ \boldsymbol{p}_{n+1} & = \boldsymbol{r} - \boldsymbol{q}_{n+1} - \boldsymbol{y}_{n+1}. \end{array} \right.$$

- Set $\boldsymbol{u}_0 = -\boldsymbol{r}$ and $(\forall n \in \mathbb{N})$ $\boldsymbol{u}_n = \boldsymbol{p}_n - \boldsymbol{r}$ and $\boldsymbol{v}_n = -\boldsymbol{q}_{n+1}$.
- Then $\boldsymbol{v}_n - \boldsymbol{u}_n = \boldsymbol{x}_n$, $\boldsymbol{v}_n - \boldsymbol{u}_{n+1} = \boldsymbol{y}_{n+1}$, and

$$\begin{cases} \boldsymbol{v}_n = \boldsymbol{p}_n - \boldsymbol{r} + \boldsymbol{x}_n = \boldsymbol{u}_n + J_{\boldsymbol{B}}(-\boldsymbol{u}_n) + \boldsymbol{b}_n \\ \boldsymbol{u}_{n+1} = \boldsymbol{p}_{n+1} - \boldsymbol{r} = -\boldsymbol{q}_{n+1} - \boldsymbol{y}_{n+1} = \boldsymbol{v}_n - J_{\boldsymbol{A}}(\boldsymbol{v}_n + \boldsymbol{r}) - \boldsymbol{a}_n. \end{cases}$$

- Set $\boldsymbol{C}\colon \boldsymbol{v} \mapsto \boldsymbol{A}^{-1}(\boldsymbol{v}+\boldsymbol{r})$ and $\boldsymbol{D} = \boldsymbol{B}^{\sim} = -\boldsymbol{B}^{-1}(-\cdot)$.
- Then $\boldsymbol{C}^{-1} = -\boldsymbol{r} + \boldsymbol{A}$, $\boldsymbol{D}^{\sim} = \boldsymbol{B}$, $J_{\boldsymbol{C}} = \mathbf{Id} - J_{\boldsymbol{A}}(\cdot + \boldsymbol{r})$, and $J_{\boldsymbol{D}} = \mathbf{Id} + J_{\boldsymbol{B}} \circ (-\mathbf{Id})$.
- Thus, $\boldsymbol{u}_0 = -\boldsymbol{r}$ and $(\forall n \in \mathbb{N})$ $\left\lfloor \begin{array}{ll} \boldsymbol{v}_n & = J_{\boldsymbol{D}}\boldsymbol{u}_n + \boldsymbol{b}_n \\ \boldsymbol{u}_{n+1} & = J_{\boldsymbol{C}}\boldsymbol{v}_n - \boldsymbol{a}_n. \end{array}\right.$
- Using [Bauschke/PLC/Reich, 2005], get

$$\boldsymbol{r} \in \operatorname{ran}(\mathbf{Id} + \boldsymbol{A} + \boldsymbol{B}) \Leftrightarrow \operatorname{Fix}(J_{\boldsymbol{C}}J_{\boldsymbol{D}}) \neq \varnothing.$$

- Deduce from Martinet's Lemma that there exists $\boldsymbol{u} \in \operatorname{Fix}(J_{\boldsymbol{C}}J_{\boldsymbol{D}})$ such that

$$\boldsymbol{x}_n = \boldsymbol{v}_n - \boldsymbol{u}_n = \boldsymbol{b}_n + J_{\boldsymbol{D}}\boldsymbol{u}_n - \boldsymbol{u}_n \to J_{\boldsymbol{D}}\boldsymbol{u} - \boldsymbol{u}.$$

- Using [Bauschke/PLC/Reich, 2005], get

$$J_{\boldsymbol{D}}\boldsymbol{u} - \boldsymbol{u} = J_{\boldsymbol{C}^{-1}+\boldsymbol{D}^{\sim}}\, 0 = J_{\boldsymbol{A}+\boldsymbol{B}}\, \boldsymbol{r}.$$

- Underlying duality: $0 \in \boldsymbol{C}\boldsymbol{x} + {}^{1}\boldsymbol{D}\boldsymbol{x} \Leftrightarrow 0 \in \boldsymbol{C}^{-1}\boldsymbol{u} + ({}^{1}\boldsymbol{D})^{\sim}\boldsymbol{u}$.

Lemma (Martinet, 1972)

Let T_1 and T_2 be firmly nonexpansive operators from $\mathcal{H}$ to $\mathcal{H}$ such that $\operatorname{Fix}(T_1 T_2) \neq \varnothing$, and let $(e_{1,n})$ and $(e_{2,n})$ be sequences in $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}} |||\boldsymbol{e}_{1,n}||| < +\infty$ and $\sum_{n\in\mathbb{N}} |||\boldsymbol{e}_{2,n}||| < +\infty$. Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be the sequence resulting from the iteration

$$\boldsymbol{u}_0 \in \mathcal{H} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \boldsymbol{u}_{n+1} = T_1\big(T_2\boldsymbol{u}_n + \boldsymbol{e}_{2,n}\big) + \boldsymbol{e}_{1,n}.$$

Then there exists $\boldsymbol{u} \in \operatorname{Fix}(T_1 T_2)$ such that $\boldsymbol{u_n} \rightharpoonup \boldsymbol{u}$. Moreover, $T_2\boldsymbol{u}_n - \boldsymbol{u}_n \to T_2\boldsymbol{u} - \boldsymbol{u}$.

Remarks

- In the case of normal cones, say $\boldsymbol{A} = N_{\boldsymbol{C}}$ and $\boldsymbol{B} = N_{\boldsymbol{D}}$, then $J_{\boldsymbol{A}} = P_{\boldsymbol{C}}$ and $J_{\boldsymbol{B}} = P_{\boldsymbol{D}}$.

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- ... but the theorem does not capture the Boyle/Dykstra theorem since $J_{N_{\boldsymbol{C}}+N_{\boldsymbol{D}}} \neq P_{\boldsymbol{C}\cap\boldsymbol{D}}$!
- In addition, how to handle $m > 2$ maximal monotone operators ?

The resolvent of the sum of m monotone operators

Theorem

Let $(A_i)_{1\leq i\leq m}$ be maximal monotone from $\mathcal{H}$ to $2^{\mathcal{H}}$. Set

$$A=\sum_{i=1}^{m}\omega_iA_i,\ \text{where } \{\omega_i\}_{1\leq i\leq m}\subset\,]0,1[\ \text{and } \sum_{i=1}^{m}\omega_i=1.$$

For every $i\in\{1,\ldots,m\}$, let $(a_{i,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}}\|a_{i,n}\|<+\infty$. Furthermore, let $r\in\operatorname{ran}(Id+A)$ and set

$$\left\lfloor\begin{array}{l} x_0=r\\ \text{For } i=1,\ldots,m\\ \quad\lfloor\ z_{i,0}=x_0 \end{array}\right. \quad\text{and}\quad (\forall n\in\mathbb{N})\ \left\lfloor\begin{array}{l} \text{For } i=1,\ldots,m\\ \quad\lfloor\ y_{i,n}=J_{A_i}z_{i,n}+a_{i,n}\\ x_{n+1}=\sum_{i=1}^{m}\omega_iy_{i,n}\\ \text{For } i=1,\ldots,m\\ \quad\lfloor\ z_{i,n+1}=x_{n+1}+z_{i,n}-y_{i,n}. \end{array}\right.$$

Then $x_n\to J_Ar$.

Proof outline

- $\boldsymbol{\mathcal{H}} = \mathcal{H}^m$ with $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \sum_{i=1}^m \omega_i \langle x_i \mid y_i \rangle$.
- $\boldsymbol{A} \colon \boldsymbol{x} \mapsto \bigtimes_{i=1}^{m} A_i x_i$.
- $\boldsymbol{B} = N_{\boldsymbol{D}}$, where $\boldsymbol{D} = \big\{(x, \ldots, x) \in \boldsymbol{\mathcal{H}} \mid x \in \mathcal{H}\big\}$.
- $\boldsymbol{j} \colon x \mapsto (x, \ldots, x)$.
- $J_{\boldsymbol{A}} \colon \boldsymbol{x} \mapsto \big(J_{A_i} x_i\big)_{1 \leq i \leq m}$ and $J_{\boldsymbol{B}} = P_{\boldsymbol{D}} \colon \boldsymbol{x} \mapsto \boldsymbol{j}\Big(\sum_{i=1}^m \omega_i x_i\Big)$.
- $\boldsymbol{j}(J_A r) = J_{\boldsymbol{A}+\boldsymbol{B}}\, \boldsymbol{j}(r)$.
- To construct $J_{\boldsymbol{A}+\boldsymbol{B}}\, \boldsymbol{j}(r)$ use Theorem 8 with $\boldsymbol{b}_n \equiv 0$.
- Since $J_{\boldsymbol{B}} = P_{\boldsymbol{D}}$, algorithm reduces to

$$\left\lfloor \begin{array}{l} \boldsymbol{y}_0 = \boldsymbol{j}(r) \\ \boldsymbol{p}_0 = 0 \end{array} \right. \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{x}_n = P_{\boldsymbol{D}}\, \boldsymbol{y}_n \\ \boldsymbol{y}_{n+1} = J_{\boldsymbol{A}}(\boldsymbol{x}_n + \boldsymbol{p}_n) + \boldsymbol{a}_n \\ \boldsymbol{p}_{n+1} = \boldsymbol{x}_n + \boldsymbol{p}_n - \boldsymbol{y}_{n+1}. \end{array} \right.$$

- After reordering and introducing $\boldsymbol{z}_n = \boldsymbol{x}_n + \boldsymbol{p}_n$:

$$\left\lfloor \begin{array}{l} \boldsymbol{x}_0 = P_{\boldsymbol{D}}\,\boldsymbol{j}(r) \\ \boldsymbol{z}_0 = \boldsymbol{x}_0 \end{array}\right. \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{y}_n = J_{\boldsymbol{A}}\boldsymbol{z}_n + \boldsymbol{a}_n \\ \boldsymbol{x}_{n+1} = P_{\boldsymbol{D}}\,\boldsymbol{y}_n \\ \boldsymbol{z}_{n+1} = \boldsymbol{x}_{n+1} + \boldsymbol{z}_n - \boldsymbol{y}_n. \end{array}\right.$$

- Now set $\boldsymbol{a}_n = (a_{i,n})_{1\leq i\leq m}$, $\boldsymbol{y}_n = (y_{i,n})_{1\leq i\leq m}$, and $\boldsymbol{z}_n = (z_{i,n})_{1\leq i\leq m}$ to get $(\forall n \in \mathbb{N})$ $\boldsymbol{x}_n = \boldsymbol{j}(x_n)$.
- Conclude that

$$x_n = \boldsymbol{j}^{-1}(\boldsymbol{x}_n) \to \boldsymbol{j}^{-1}(J_{\boldsymbol{A}+\boldsymbol{B}}\,\boldsymbol{j}(r)) = J_A r.$$

The proximity operator of the sum

- Let $(f_i)_{1\leq i\leq m}$ be functions in $\Gamma_0(\mathcal{H})$ such that $\bigcap_{i=1}^m \operatorname{dom} f_i \neq \varnothing$.
- Set $f = \sum_{i=1}^m \omega_i f_i$, where $\{\omega_i\}_{1\leq i\leq m} \subset \,]0,1[$ and $\sum_{i=1}^m \omega_i = 1$.
- For every $r \in \mathcal{H}$,

$$\operatorname{prox}_f r = \operatorname{argmin}_{x\in\mathcal{H}} f(x) + \frac{1}{2}\|r-x\|^2$$

  is uniquely defined.
- Setting $A_i = \partial f_i$ in the theorem and assuming some CQ so that $\partial f = \sum_{i=1}^m \omega_i A_i$, we can construct $\operatorname{prox}_f r$.

The prox of the sum of two convex functions

Proposition [Bauschke/PLC/Reich, 2005]

Let $(\mathcal{H}, |||\cdot|||)$ be a real Hilbert space, and let φ and ψ be functions in $\Gamma_0(\mathcal{H})$ such that

$$\inf \; \varphi + \operatorname{env}(\psi) > -\infty.$$

Set

$$\boldsymbol{u}_0 \in \mathcal{H} \text{ and } (\forall n \in \mathbb{N}) \; \boldsymbol{v}_n = \operatorname{prox}_{\psi} \boldsymbol{u}_n \text{ and } \boldsymbol{u}_{n+1} = \operatorname{prox}_{\varphi} \boldsymbol{v}_n.$$

Then $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$, where $\boldsymbol{u} \in \operatorname{argmin} \varphi + \operatorname{env}(\psi)$, and $\operatorname{prox}_{\psi} \boldsymbol{u}_n - \boldsymbol{u}_n \to \boldsymbol{w}$, where $\boldsymbol{w} = \operatorname{prox}_{\varphi^* + \psi^{*\vee}} 0$ is the unique solution to the dual problem

$$\inf \; \varphi^* + \psi^{*\vee} + \frac{1}{2} |||\cdot|||^2.$$

The prox of the sum of two convex functions

Theorem (Bauschke/PLC, 2008)

Let $(\mathcal{H}, |||\cdot|||)$ *be a real Hilbert space, and let* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *be functions in* $\Gamma_0(\mathcal{H})$ *such that* $\operatorname{dom}\boldsymbol{f} \cap \operatorname{dom}\boldsymbol{g} \neq \varnothing$*. Furthermore, let* $\boldsymbol{r} \in \mathcal{H}$ *and let* $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ *be the sequence generated by the following routine.*

$$\left\lfloor \begin{array}{l} \boldsymbol{y}_0 = \boldsymbol{r} \\ \boldsymbol{p}_0 = 0 \\ \boldsymbol{q}_0 = 0 \end{array} \right. \quad \text{and} \quad \left\lfloor \begin{array}{ll} \boldsymbol{x}_n & = \operatorname{prox}_{\boldsymbol{g}}(\boldsymbol{y}_n + \boldsymbol{q}_n) \\ \boldsymbol{q}_{n+1} & = \boldsymbol{y}_n + \boldsymbol{q}_n - \boldsymbol{x}_n \\ \boldsymbol{y}_{n+1} & = \operatorname{prox}_{\boldsymbol{f}}(\boldsymbol{x}_n + \boldsymbol{p}_n) \\ \boldsymbol{p}_{n+1} & = \boldsymbol{x}_n + \boldsymbol{p}_n - \boldsymbol{y}_{n+1}. \end{array} \right.$$

Then $\boldsymbol{x}_n \to \operatorname{prox}_{\boldsymbol{f}+\boldsymbol{g}}\, \boldsymbol{r}$.

Proof highlights

- Set $\varphi \colon \boldsymbol{v} \mapsto \boldsymbol{f}^*(\boldsymbol{v}+\boldsymbol{r}) - \frac{1}{2}\|\boldsymbol{r}\|^2$ and $\psi = \boldsymbol{g}^{*\vee}$.
- Then $\operatorname{prox}_{\varphi}\boldsymbol{v} = \boldsymbol{v} - \operatorname{prox}_{\boldsymbol{f}}(\boldsymbol{v}+\boldsymbol{r})$ and $\operatorname{prox}_{\psi}\boldsymbol{v} = \boldsymbol{v} + \operatorname{prox}_{\boldsymbol{g}}(-\boldsymbol{v})$.
- Use Fenchel, some changes of variables, algebraic manipulations, and the proposition on the dual asymptotic behavior of the alternating prox algorithm.

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- Set $\varphi\colon \boldsymbol{v} \mapsto \boldsymbol{f}^*(\boldsymbol{v}+\boldsymbol{r}) - \frac{1}{2}\|\boldsymbol{r}\|^2$ and $\psi = \boldsymbol{g}^{*\vee}$.
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- Use Fenchel, some changes of variables, algebraic manipulations, and the proposition on the dual asymptotic behavior of the alternating prox algorithm.

Remark

When $\boldsymbol{f} = \iota_{\boldsymbol{C}}$ and $\boldsymbol{g} = \iota_{\boldsymbol{D}}$ we do recover the Boyle/Dykstra theorem.

The prox of the sum of m convex functions

Proposition

Let $(f_i)_{1\leq i\leq m}$ be functions in $\Gamma_0(\mathcal{H})$ such that $\bigcap_{i=1}^m \operatorname{dom} f_i \neq \varnothing$. Set

$$f = \sum_{i=1}^m \omega_i f_i, \text{ where } \{\omega_i\}_{1\leq i\leq m} \subset \left]0,1\right[\text{ and } \sum_{i=1}^m \omega_i = 1.$$

Furthermore, let $r \in \mathcal{H}$ and set

$$\left\lfloor \begin{array}{l} x_0 = r \\ \text{For } i = 1,\ldots,m \\ \quad \lfloor\; z_{i,0} = x_0 \end{array} \right. \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \text{For } i = 1,\ldots,m \\ \quad \lfloor\; y_{i,n} = \operatorname{prox}_{f_i} z_{i,n} \\ x_{n+1} = \sum_{i=1}^m \omega_i y_{i,n} \\ \text{For } i = 1,\ldots,m \\ \quad \lfloor\; z_{i,n+1} = x_{n+1} + z_{i,n} - y_{i,n}. \end{array} \right.$$

Then $x_n \to \operatorname{prox}_f r$.

The prox of the sum of m convex functions

Proposition

Let $(f_i)_{1\leq i\leq m}$ be functions in $\Gamma_0(\mathcal{H})$ such that $\bigcap_{i=1}^m \operatorname{dom} f_i \neq \varnothing$. Set

$$f = \sum_{i=1}^m \omega_i f_i, \ \text{where } \{\omega_i\}_{1\leq i\leq m} \subset \,]0,1[\ \text{and} \sum_{i=1}^m \omega_i = 1.$$

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Then $x_n \to \operatorname{prox}_f r$.

Remark

The above result provides a strongly convergent, qualification-free minimization algorithm for strongly convex problems.

Projecting onto the intersection of convex sets

Corollary (Gaffke/Mathar, 1989)

Let $(C_i)_{1\leq i\leq m}$ be closed convex subsets of $\mathcal{H}$ such that $C=\bigcap_{i=1}^m C_i\neq\varnothing$. Take $\{\omega_i\}_{1\leq i\leq m}\subset\,]0,1[$ such that $\sum_{i=1}^m\omega_i=1$. Furthermore, let $r\in\mathcal{H}$ and set

$$\left\lfloor\begin{array}{l} x_0 = r\\ \text{For } i=1,\ldots,m\\ \lfloor\ z_{i,0}=x_0\end{array}\right. \quad\text{and}\quad (\forall n\in\mathbb{N})\quad \left\lfloor\begin{array}{l}\text{For } i=1,\ldots,m\\ \lfloor\ y_{i,n}=P_{C_i}z_{i,n}\\ x_{n+1}=\sum_{i=1}^m\omega_i y_{i,n}\\ \text{For } i=1,\ldots,m\\ \lfloor\ z_{i,n+1}=x_{n+1}+z_{i,n}-y_{i,n}.\end{array}\right.$$

Then $x_n\to P_C r$.

Remark (Lapidus, 1980)

Suppose that the sets $(C_i)_{1\leq i\leq m}$ are closed vector subspaces. Then the update rule reduces to $x_{n+1}=\sum_{i=1}^m\omega_i P_{C_i}x_n$. Hence, $(\sum_{i=1}^m\omega_i P_{C_i})^n\to P_C$.

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