Some new results in convex subdifferential calculus

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1. Introduction

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   a. Subdifferential for the sum function.
   b. Extension of Hiriart-Urruty–Phelps formula.
   c. Chain rule under the Moreau–Rockafellar constraint qualification.

2. Notations and basic tools

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θ : zero in all the involved spaces.

$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

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$$A + B := \{a + b \mid a \in A, b \in B\}, \quad A + \emptyset := \emptyset + A := \emptyset;$$

and, if $\emptyset \neq \Lambda \subset \mathbb{R}$ we set

$$\Lambda A := \{\lambda a \mid \lambda \in \Lambda,\ a \in A\}, \quad \Lambda\emptyset := \emptyset.$$

Furthermore, $\Lambda x := \Lambda\{x\}$, $\lambda A := \{\lambda\}A$ and $x + A := \{x\} + A$.

$\text{co}\, A$: *convex hull* of A,

$\text{cone}\, A$: *conic hull* of A,

$\text{aff}\, A$: *affine hull* of the set A,

$\operatorname{co} A$: *convex hull* of A,

$\operatorname{cone} A$: *conic hull* of A,

$\operatorname{aff} A$: *affine hull* of the set A,

$\operatorname{int} A$: *interior* of A,

$\operatorname{cl} A$ and $\overline{A}$: *closure* of A (w^*–*closure* if $A \subset X^*$).

We set $\overline{\operatorname{co}}A := \operatorname{cl}(\operatorname{co} A)$ and $\overline{\operatorname{cone}}A := \operatorname{cl}(\operatorname{cone} A)$.

$\operatorname{ri} A$: topological *relative interior* of A (i.e., the interior of A in the topology relative to $\operatorname{aff} A$ if $\operatorname{aff} A$ is closed, and the empty set otherwise).

Associated with $A \neq \emptyset$ we consider the sets

$$A^\circ := \{x^* \in X^* \mid \langle x, x^* \rangle \geq -1 \ \forall x \in A\},$$
$$A^- := -(\operatorname{cone} A)^\circ = \{x^* \in X^* \mid \langle x, x^* \rangle \leq 0 \ \forall x \in A\},$$
$$A^\perp := (-A^-) \cap A^- = \{x^* \in X^* \mid \langle x, x^* \rangle = 0 \ \forall x \in A\},$$

i.e. the (one-sided) *polar*, the *negative dual cone*, and the *orthogonal subspace* (or *annihilator*) of A, respectively.

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i.e. the (one-sided) *polar*, the *negative dual cone*, and the *orthogonal subspace* (or *annihilator*) of A, respectively.

By the *bipolar theorem* , we have

$$A^{--} = \overline{\operatorname{cone}}(\operatorname{co} A).$$

If $A \subset X$ is convex and $x \in X$, we define the *normal cone* to A at x as

$$\mathrm{N}_A(x) := \begin{cases} (A - x)^- & \text{if } x \in A, \\ \emptyset & \text{if } x \in X \setminus A. \end{cases}$$

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If $A \neq \emptyset$ is convex and closed, A_∞ represents its *recession cone* defined as

$$A_\infty := \{y \in X \mid x + \lambda y \in X \text{ for some } x \in X \text{ and } \forall \lambda \geq 0\}.$$

Given a function $h : X \longrightarrow \overline{\mathbb{R}}$, its *(effective) domain* and *epigraph* are defined by

$$\operatorname{dom} h := \{x \in X \mid h(x) < +\infty\},$$
$$\operatorname{epi} h := \{(x, \alpha) \in X \times \mathbb{R} \mid h(x) \leq \alpha\}.$$

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h is *proper* if $\operatorname{dom} h \neq \emptyset$ and $h(x) > -\infty$ for all $x \in X$. Then we consider the *graph* of h which is defined by

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h is *convex* if $\operatorname{epi} h$ is convex.

The *lower closure* of h is the function $\mathrm{cl}\, h : X \longrightarrow \overline{\mathbb{R}}$ defined by

$$(\mathrm{cl}\, h)(x) := \inf\{t \mid (x,t) \in \mathrm{cl}(\mathrm{epi}\, h)\}.$$

We have:

- $\mathrm{epi}\,(\mathrm{cl}\, h) = \mathrm{cl}\,(\mathrm{epi}\, h)$. Then, $\mathrm{cl}\, h$ is the *greatest* lower semicontinuous (lsc, in brief) function dominated by h; i.e. $\mathrm{cl}\, h \leq h$.

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- If h is convex, then $\operatorname{cl} h$ is also convex, and $\operatorname{cl} h$ does not take the value $-\infty$ iff h admits a continuous affine minorant.
- Given $h : X \longrightarrow \overline{\mathbb{R}}$, the *lsc convex hull* of h is the lsc convex function $\overline{\operatorname{co}} h : X \longrightarrow \overline{\mathbb{R}}$ such that $\operatorname{epi}(\overline{\operatorname{co}} h) = \overline{\operatorname{co}}(\operatorname{epi} h)$.
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$\Lambda(X)$: set of all the proper convex functions on X
$\Gamma(X)$: subset of $\Lambda(X)$ consisting of the lsc functions.

The *Legendre-Fenchel conjugate* of h is the lsc convex function $h^* : X^* \longrightarrow \overline{\mathbb{R}}$ given by

$$h^*(x^*) := \sup\{\langle x, x^* \rangle - h(x) \mid x \in X\}.$$

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The *bi-conjugate* of h is the function $h^{**} : X \longrightarrow \overline{\mathbb{R}}$ given by

$$h^{**}(x) := \sup\{\langle x, x^* \rangle - h^*(x^*) \mid x^* \in X^*\}.$$

We have

$$\{h \in \overline{\mathbb{R}}^X : \ h = h^{**}\} = \Gamma(X) \cup \{+\infty\}^X \cup \{-\infty\}^X.$$

Moreover, $h^{**} \leq \overline{\operatorname{co}} h$, and the equality holds if h admits a continuous affine minorant.

The *support* and the *indicator* functions of $A \neq \emptyset$ are defined as

$$\sigma_A(x^*) := \sup\{\langle a, x^* \rangle \mid a \in A\}, \text{ for } x^* \in X^*, \text{ and}$$

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$$\partial_\varepsilon h(x) := \{x^* \in X^* \mid h(y) - h(x) \geq \langle y - x, x^* \rangle - \varepsilon \; \forall y \in X\}.$$

If $h(x) \notin \mathbb{R}$ we set $\partial_\varepsilon h(x) := \emptyset$. In particular, for $\varepsilon = 0$ we get $\partial h(x) := \partial_0 h(x)$, the *subdifferential* of h at x.
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c) If h is convex, then

$$\partial_\varepsilon h(x) \neq \emptyset \; \forall \varepsilon > 0 \Longleftrightarrow h \text{ is lsc at } x.$$

d) If A is convex and $x \in A$,

$$\partial \mathrm{I}_A(x) = (\mathrm{cone}(A - x))^- = \mathrm{N}_A(x).$$

3. Optimal set for the *relaxed problem*

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For convenience we set $\varepsilon -$ argmin $h = \emptyset$ for all $\varepsilon \geq 0$ whenever $m \notin \mathbb{R}$.

Next we establish the main result in this section.

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Theorem 1

For any function $h : X \to \overline{\mathbb{R}}$ *such that* $\operatorname{dom} h^* \neq \emptyset$, *one has*

$$\operatorname{argmin} h^{**} = \bigcap_{\substack{\varepsilon>0 \\ x^* \in \operatorname{dom} h^*}} \overline{\operatorname{co}}\left((\varepsilon - \operatorname{argmin} h) + \{x^*\}^-\right).$$

If $\operatorname{cone}(\operatorname{dom} h^*)$ *is* w^**−closed or* $\operatorname{ri}(\operatorname{cone}(\operatorname{dom} h^*)) \neq \emptyset$, *then*

$$\operatorname{argmin} h^{**} = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left((\varepsilon - \operatorname{argmin} h) + (\operatorname{dom} h^*)^-\right).$$

In particular, if $\operatorname{cone}(\operatorname{dom} h^*)) = X^*$ *, then*

$$\operatorname{argmin} h^{**} = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left(\varepsilon - \operatorname{argmin} h\right).$$

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the inverse multivalued mapping of the $\varepsilon-$subdifferential of h. In other words, for any $x^* \in X^*$,

$$\begin{aligned} (M_\varepsilon h)(x^*) &= (\partial_\varepsilon h)^{-1}(x^*) = \{x \in X : \ x^* \in \partial_\varepsilon h(x)\} \\ &= \{x \in X : \ h(x) - \langle x, x^* \rangle \leq -h^*(x^*) + \varepsilon\} \\ &= \varepsilon - \operatorname{argmin}\left(h(\cdot) - \langle \cdot, x^* \rangle\right). \end{aligned}$$

We are now in position to state the following result:

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Theorem 2

For any function $h : X \to \overline{\mathbb{R}}$ *such that* $\operatorname{dom} h^* \neq \emptyset$*, one has for all* $x^* \in X^*$*,*

$$\partial h^*(x^*) = \bigcap_{\substack{\varepsilon>0 \\ u^* \in \operatorname{dom} h^*}} \overline{\operatorname{co}}\left((M_\varepsilon h)(x^*) + \{u^* - x^*\}^-\right).$$

If $\operatorname{cone}\left((\operatorname{dom} h^*) - x^*\right)$ *is* w^**−closed or* $\operatorname{ri}(\operatorname{cone}((\operatorname{dom} h^*) - x^*)) \neq \emptyset$*, then*

$$\partial h^*(x^*) = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left((M_\varepsilon h)(x^*) + \mathrm{N}_{\operatorname{dom} h^*}(x^*)\right).$$

Theorem 3

Given $\{f_t,\ t \in T\} \subset \overline{\mathbb{R}}^X$, $T \neq \emptyset$, *consider the supremum function* $f := \sup_{t\in T} f_t$. *Assume that* $\operatorname{dom} f \neq \emptyset$ *and that*

$$f^{**} \equiv \left(\sup_{t\in T} f_t\right)^{**} = \sup_{t\in T} f_t^{**}.$$

Then, at every $x \in X$, *we have*

$$\partial f(x) = \bigcap_{\varepsilon>0,\ z\in \operatorname{dom} f} \overline{\operatorname{co}} \left(\bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \{z - x\}^- \right),$$

where $T_\varepsilon(x) := \{t \in T :\ f_t(x) \geq f(x) - \varepsilon\}$ *if* $f(x) \in \mathbb{R}$ *and* $T_\varepsilon(x) = \emptyset$ *if* $f(x) \notin \mathbb{R}$.

Theorem 3

If, moreover, $\operatorname{cone}\operatorname{co}(\operatorname{dom} f - x)$ *is closed or* $\operatorname{ri}(\operatorname{cone}\operatorname{co}(\operatorname{dom} f - x)) \neq \emptyset$*, then*

$$\partial f(x) = \bigcap_{\varepsilon>0} \overline{\operatorname{co}} \left(\bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \mathrm{N}_{\operatorname{dom} f}(x) \right).$$

Theorem 4

Given $h : X \to \overline{\mathbb{R}}$ *and* $\{C_i,\ i \in I\}$, *convex subsets of* X^* *satisfying*

$$\operatorname{dom} h^* \subseteq \bigcup_{i \in I} C_i,$$

and

$$\operatorname{ri}\left(\operatorname{cone}(C_i \cap \operatorname{dom} h^*)\right) \neq \emptyset, \text{ for all } i \in I,$$

one has

$$\operatorname{argmin}\ h^{**} = \bigcap_{\varepsilon>0,\ i \in I} \overline{\operatorname{co}}\left((\varepsilon - \operatorname{argmin} h) + (C_i \cap \operatorname{dom} h^*)^-\right).$$

Remark: If we take $\{C_i,\ i \in I\} = \{\{x^*\},\ x^* \in \operatorname{dom} h^*\}$, we get Theorem 1.

Corollary 1

For any function $h : X \to \overline{\mathbb{R}}$ *with* $\operatorname{dom} h^* \neq \emptyset$, *if*

$$\mathcal{F}_{x^*} := \left\{ L \subset X^* \;\middle|\; \begin{array}{c} L \text{ is a finite-dimensional linear subspace} \\ \text{such that } x^* \in L \end{array} \right\},$$

one has for all $x^* \in X^*$,

$$\partial h^*(x^*) = \bigcap_{\varepsilon > 0,\, L \in \mathcal{F}_{x^*}} \overline{\operatorname{co}} \left((M_\varepsilon h)(x^*) + \mathrm{N}_{L \cap \operatorname{dom} h^*}(x^*) \right).$$

Remember that $(M_\varepsilon h)(x^*) = \varepsilon - \operatorname{argmin} \left(h(\cdot) - \langle \cdot, x^* \rangle \right).$

4. Subdifferential of the supremum function

Theorem 5 (HLV09)

Given nonempty family $\{f_t,\ t \in T\} \subset \overline{\mathbb{R}}^X$, *consider the supremum function* $f := \sup_{t\in T} f_t$, *and assume that* $\operatorname{dom} f \neq \emptyset$ *and*

$$f^{**} = \sup_{t\in T} f_t^{**}. \tag{CC}$$

Then, for every $x \in X$,

$$\partial f(x) = \bigcap\nolimits_{\varepsilon>0,\ L\in\mathcal{F}_x} \operatorname{cl}\left(\operatorname{co}\left(\bigcup\nolimits_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x)\right) + \mathrm{N}_{L\cap \operatorname{dom} f}(x)\right).$$

The following lemma provides alternative characterizations of $N_{\operatorname{dom} f}(z)$.

Lemma 1

Let $T \neq \emptyset$, $\{f_t,\ t \in T\} \subset \overline{\mathbb{R}}^X$, *and* $f := \sup\{f_t : t \in T\}$. *Then, for every* $x \in \operatorname{dom} f$ *we have*

$$\begin{aligned} x^* \in N_{\operatorname{dom} f}(x) &\Longleftrightarrow (x^*, \langle x^*, x\rangle) \in [\overline{\operatorname{co}}\,(\cup_{t\in T} \operatorname{gph} f_t^*)]_\infty \\ &\Longleftrightarrow (x^*, \langle x^*, z\rangle) \in [\overline{\operatorname{co}}\,(\cup_{t\in T} \operatorname{epi} f_t^*)]_\infty \\ &\Longleftrightarrow (x^*, \langle x^*, z\rangle) \in (\operatorname{epi} f^*)_\infty \\ &\Longleftrightarrow (x^*, \langle x^*, z\rangle) \in \operatorname{epi}(\sigma_{\operatorname{dom} f}). \end{aligned}$$

In the affine case our formula takes a simpler form:

Corollary 1

Assume that $T \neq \emptyset$ *and* $f(x) := \sup\{\langle a_t^*, x\rangle - \beta_t \mid t \in T\}$, *with* $a_t^* \in X^*$ *and* $\beta_t \in \mathbb{R}$. *Then, for every* $x \in X$ *we have*

$$\partial f(x) = \bigcap_{L \in \mathcal{F}_x, \varepsilon > 0} \mathrm{cl}\left(\mathrm{co}\{a_t^* \mid t \in T_\varepsilon(x)\} + B_L\right),$$

where

$$x^* \in B_L \Leftrightarrow (x^*, \langle x^*, x\rangle) \in \left[\overline{\mathrm{co}}\left((L^\perp \times \{0\}) \cup \{(a_t^*, \beta_t), t \in T\}\right)\right]_\infty.$$

Corollary 2

Let $\{f_t : X \to \overline{\mathbb{R}} \mid t \in T\}$ *be a non-empty family of convex functions and set* $f := \sup_{t\in T} f_t$. *Assume that one of the following conditions holds:*

(1) - All the functions f_t *with* $t \in T$ *are lsc.*

(2) - $\exists x_0 \in \operatorname{dom} f$ *such that* f_t *is continuous at* x_0, $\forall t \in T$.

(3) - $T := \{1, \ldots, k, k+1\}$ *and* $\exists x_0 \in \operatorname{dom} f_{k+1} \cap (\bigcap_{i=1}^{k} \operatorname{dom} f_i)$ *such that* $f_1, \ldots, f_k$ *are continuous at* x_0.

(4) - $X = \mathbb{R}^n$ *and* $\operatorname{dom} f \cap (\cap_{t\in T} \operatorname{ri}(\operatorname{dom} f_t))$ *is nonempty.*

Then, (CC) holds and for every $x \in X$

$$\partial f(x) = \bigcap_{L\in\mathcal{F}_x,\ \varepsilon>0} \operatorname{cl}\left(\operatorname{co}\left(\bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x)\right) + \mathrm{N}_{L\cap \operatorname{dom} f}(x)\right).$$

The following result, due to Volle (see, e.g., [22]), is originally established in the context of normed spaces.

Corollary 3

Let $\{f_t : X \to \overline{\mathbb{R}} \mid t \in T\}$ be a non-empty family of convex functions, and set $f := \sup_{t\in T} f_t$. Assume that f is finite and continuous at $z \in X$. Then, we have

$$\partial f(z) = \bigcap_{\varepsilon>0} \overline{\text{co}} \left(\bigcup_{t\in T_\varepsilon(z)} \partial_\varepsilon f_t(z)\right).$$

Proof. Because f is finite and continuous at z we have that $z \in \text{int}(\text{dom} f)$, and so $\text{N}_{\text{dom} f}(z) = \{\theta\}$. Further, as $z \in \cap_{t\in T} \text{int}(\text{dom} f_t)$, Condition (2) of Corollary 2 yields $\text{cl} f = \sup\{\text{cl} f_t \mid t \in T\}$, and so the conclusion follows. ■

Using our approach we derive the following result which is due to Brøndsted (e.g., [1]); see also [3, Proposition 7], where such a formula is extended to families of infinitely many convex functions defined on $\mathbb{R}^n$.

Corollary 4

Consider the convex functions $f_i : X \to \overline{\mathbb{R}}$ for $i = 1, \ldots, k$, and set $f := \max\{f_1, \ldots, f_k\}$. Assume that

$$\mathrm{cl} f = \max\{\mathrm{cl} f_1, \ldots, \mathrm{cl} f_k\}.$$

Given $z \in X$ such that $(\mathrm{cl} f)(z) = (\mathrm{cl} f_i)(z)$ for $i = 1, \ldots, k$, we have

$$\partial f(z) = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}} \left(\bigcup_{i=1}^{k} \partial_\varepsilon f_i(z) \right).$$

4. Other calculus rules

- X, Y (separated) real locally convex spaces
- $f : Y \to \overline{\mathbb{R}}$, $g : X \to \overline{\mathbb{R}}$ convex functions
- $A : X \to Y$ a continuous affine mapping

$$Ax = A_0 x + b,$$

where A_0 is the linear part of A and $b \in Y$. We denote by A_0^* the adjoint operator of A_0.
We show that our formula for the subdifferential of the supremum function also gives calculus rules for other operations, expressed by means of the convex function $g + f \circ A$.
The resulting formulas are not new, but our aim here is to highlight the unifying character of Theorem 5.

We derive a slight extension of Hiriart-Urruty –Phelps formula [6] . This allows us to express $\partial(g + f \circ A)$ in terms of the approximate subdifferentials of f and g.

Theorem 6

Under the current notation, assume that the following holds

$$\operatorname{cl}(g + f \circ A) = (\operatorname{cl} g) + (\operatorname{cl} f) \circ A.$$

Then, for every $z \in X$ we have

$$\partial(g + f \circ A)(z) = \bigcap_{\varepsilon > 0} \operatorname{cl}\left(\partial_\varepsilon g(z) + A_0^* \partial_\varepsilon f(Az)\right).$$

Assuming that f and g are lsc in Theorem 3 we obtain the following result of Hiriart-Urruty–Phelps:

Corollary 5

Let f, g, and A be as in Theorem 3. If f and g are lsc , then for every $z \in X$ we have

$$\partial(g + f \circ A)(z) = \bigcap_{\varepsilon > 0} \operatorname{cl}\left(\partial_\varepsilon g(z) + A_0^* \partial_\varepsilon f(Az)\right).$$

Lemma 3

Let $f : Y \to \overline{\mathbb{R}}$ and $g : X \to \overline{\mathbb{R}}$ be convex functions and $A : X \to Y$ be a continuous affine mapping. Assume that f is finite and continuous at Ax_0 for some $x_0 \in (\operatorname{dom} g) \cap A^{-1} (\operatorname{dom} f)$. Then

$$\operatorname{cl}(f \circ A + g) = (\operatorname{cl} f) \circ A + (\operatorname{cl} g).$$

Corollary 6

([13], p. 47) Under the assumptions of Lemma 3 and denoting by A_0 the linear part of A, we have, for every $z \in X$,

$$\partial(f \circ A + g)(z) = A_0^* \partial f(Az) + \partial g(z).$$

References

[1] A. Brøndsted: On the subdifferential of the supremum of two convex functions, *Math. Scand.*, **31** (1972), 225–230.

[2] A. Hantoute: Subdifferential set of the supremum of lower semi-continuous convex functions and the conical hull property, *Top*, **14** (2006), 355–374.

[3] A. Hantoute, M.A. López: A complete characterization of the subdifferential set of the supremum of an arbitrary family of convex functions, *J. Convex Anal.*, **15** (2008), 831-858.

[4] A. Hantoute, M.A. López, C. Zălinescu: Subdifferential calculus rules in convex analysis: A unifying approach via pointwise supremum functions, *SIAM J. Optim.* **19** (2008), 863-882.

- [5] J.-B. Hiriart-Urruty, R.R. Phelps: Subdifferential calculus using ε-subdifferentials, *J. Funct. Anal.*, **118** (1993), 154–166.
- [6] J.-B. Hiriart-Urruty, M. Moussaoui, A. Seeger, M. Volle: Subdifferential calculus without qualification conditions, using approximate subdifferentials: A survey, *Nonlinear Anal.*, **24** (1995), 1727–1754.
- [7] A.D. Ioffe, V.L. Levin: Subdifferentials of convex functions, *Trudy Moskov Mat. Obshch*, **26** (1972) 3–73 (Russian).
- [8] A.D. Ioffe, V.H. Tikhomirov: *Theory of Extremal Problems*, Studies in Mathematics and its Applications, Vol. 6, North-Holland, Amsterdam, 1979.
- [9] F. Jules, M. Lassonde: Formulas for subdifferentials of sums of convex functions, *J. Convex Anal.*, **9** (2002), 519–533.

- [10] V.L. Levin: An application of Helly's theorem in convex programming, problems of best approximation and related questions. *Mat. Sb., Nov. Ser.* **79**(121) (1969), 250–263. English transl.: Math. USSR, Sb. 8, 235-247.
- [11] J.-J. Moreau: *Fonctionnelles convexes*, Rome: Instituto Poligrafico e Zecca dello Stato, 2003.
- [12] J.-P. Penot: Subdifferential calculus without qualification assumptions, *J. Convex Anal.*, **3** (1996), 207–219.
- [13] R.R. Phelps: *Convex functions, monotone operators and differentiability*, 2nd ed., Lecture Notes in Mathematics, Vol 1364, Springer-Verlag, Berlin (1993).

- [14] B.N. Pschenichnyi: Convex programming in a normalized space, *Kibernetika*, **5** (1965), 46–54 (Russian); translated as *Cybernetics* 1 (1965) no. 5, 46–57 (1966).
- [15] R.T. Rockafellar: Directionally Lipschitzian functions and subdifferential calculus, *Proc. London Math. Soc.*, **39** (1979), 331–355.
- [16] R.T. Rockafellar: *Convex Analysis*, Princeton University Press, Princeton, N.J., 1970.
- [17] R.T. Rockafellar: *Conjugate Duality and Optimization*, in: CBMS Regional Conference Series in Applied Mathematics **16**, SIAM VI, Philadelphia, Pa. (1974).
- [18] V.N. Solov'ev: The subdifferential and the directional derivatives of the maximum of a family of convex functions, *Izvestiya RAN: Ser. Mat.*, **65** (2001), 107–132.

- [19] L. Thibault: Sequential convex subdifferential calculus and sequential Lagrange multipliers, *SIAM J. Control Optim.*, **35** (1997), 1434–1444.
- [20] V.M. Tikhomirov: *Analysis II, Convex Analysis and Approximation Theory*, RX Gamkrelidze (Ed.), Encyclopedia of Mathematics Vol 14 (1987).
- [21] M. Valadier: Sous-différentiels d'une borne supérieure et d'une somme continue de fonctions convexes, C. R. Acad. Sci. Paris Sér. A-B **268** (1969), A39–A42.
- [22] M. Volle: Sous-différentiel d'une enveloppe supérieure de fonctions convexes, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), 845–849.

[23] M. Volle: On the subdifferential of an upper envelope of convex functions, *Acta Math. Vietnam.*, **19** (1994), 137–148.

[24] C. Zalinescu: Stability for a class of nonlinear optimization problems and applications, in Nonsmooth optimization and related topics (Erice, 1988), 437–458, Ettore Majorana Internat. Sci. Ser. Phys. Sci., 43, Plenum, New York, 1989.

[25] C. Zalinescu: On several results about convex set functions, *J. Math. Anal. Appl.*, **328** (2007), 1451–1470.

[26] C. Zalinescu: *Convex Analysis in General Vector Spaces*, World Scientific, Singapore, 2002.

New references:

- J.-B. Hiriart-Urruty, M.A. López, M. Volle: The $\varepsilon-$strategy in variational analysis, 2009.
- M.A. López, M. Volle: A formula for the set of optimal solutions of a relaxed minimization problem. Applications to subdifferential calculus, *J. Convex Anal.*, to appear.