THE JORDAN-VON NEUMANN CONSTANT AND FIXED POINTS FOR MULTIVALUED NONEXPANSIVE MAPPINGS*

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ABSTRACT. The purpose of this paper is to study the existence of fixed point for nonexpansive multivalued mappings in a particular class of Banach spaces. Furthermore, we demonstrate a relationship between the weakly convergent sequence coefficient $WCS(X)$ and the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ of a Banach space X. Using this fact, we prove that if $C_{\mathrm{NJ}}(X)$ is less than an appropriate positive number, then every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty weakly compact convex subset of a Banach space X, and $KC(E)$ is the class of all nonempty compact convex subsets of E.

Keywords: Multivalued nonexpansive mapping; Banach space; weakly convergent sequence coefficient; Jordan-von Neumann constant; normal structure; regular asymptotically uniform sequence; Property (D).

1. INTRODUCTION

In 1969, Nadler [**?**] established the multivalued version of Banach's contraction principle. Since then the metric fixed point theory of multivalued mappings has been rapidly developed. Some classical fixed point theorems for singlevalued nonexpansive mappings have been extended to multivalued nonexpansive mappings. In 1974, Lim [**?**], using Edelstein's method of asymptotic center, proved the existence of a fixed point for a multivalued nonexpansive self-mapping $T : E \to K(E)$ where E is a nonempty bounded closed convex subset of a uniformly convex Banach space X. In 1990, Kirk and Massa [**?**] extended Lim's theorem. They proved that every multivalued nonexpansive self-mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a Banach space X for which the asymptotic center in E of each bounded sequence of X is nonempty and compact. In 2001, Xu [**?**] extended Kirk-Massa's theorem to nonself-mapping $T : E \to KC(X)$ which satisfies the inwardness condition.

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In 2004, Dominguez and Lorenzo [?] proved that every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a Banach space X with $\varepsilon_\beta(X) < 1$. Consequently, they can give an affirmative answer of a problem in [?] proving that every nonexpansive self-mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a nearly uniformly convex Banach space. Recently, Dhompongsa at el. [?], gave an existence of a fixed point for a multivalued nonexpansive and $1-\chi-$contractive mapping $T : E \to KC(X)$ such that $T(E)$ is a bounded set and which satisfies the inwardness condition, where E is a nonempty bounded closed convex separable subset of a reflexive Banach space which satisfies the Dominguez-Lorenzo condition, i.e., an inequality concerning the asymptotic radius and the Chebyshev radius of the asymptotic center for some types of sequences. Consequently, they could show that if X is a uniformly nonsquare Banach space satisfying property WORTH and $T : E \to KC(X)$ is a nonexpansive mapping such that $T(E)$ is a bounded set and which satisfies the inwardness condition, where E is a nonempty bounded closed convex separable subset of X, then T has a fixed point. Furthermore, they also ask : Does $C_{\rm NJ}(X) < \frac{1+\sqrt{3}}{2}$ imply the existence of a fixed point for multivalued nonexpansive mappings ?

In this paper, we organize as follows. We define a property for a Banach spaces which we call property (D) (see definition in Section 3), which is weaker than the Dominguez-Lorenzo condition and stronger than weak normal structure and we prove that if X is a Banach space satisfying property (D) and E is a nonempty weakly compact convex subset of X, then every nonexpansive mapping $T : E \to KC(E)$ has a fixed point. Then we state a relationship between the weakly convergent sequence coefficient $WCS(X)$ and the Jordan-von Neumann constant $C_{\rm NJ}(X)$ of a Banach space X. Finally, using this fact, we prove that if $C_{\rm NJ}(X)$ is less than an appropriate positive number, then every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point. In particular, we give a partial answer to the question which has been asked in [?].

2. Preliminaries

Let X be a Banach space and E a nonempty subset of X. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of E, by $K(E)$ the family of nonempty compact subsets of E, and by $KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A,B) = \max\left\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\right\}, \quad A,B \in FB(X),$$

where $\operatorname{dist}(a,B) = \inf\{\|a-b\| : b \in B\}$ is the distance from the point a to the subset B. A multivalued mapping $T : E \to FB(X)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that

$$H(Tx,Ty) \le k\|x-y\|, \quad x,y \in E. \tag{2.1}$$

If (**??**) is valid when $k = 1$, then T is called nonexpansive. A point x is a fixed point for a multivalued mapping T if $x \in Tx$.

Throughout the paper we let X^* stand for the dual space of a Banach space X. By B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively. Let A be a nonempty bounded subset of X. The number $r(A) = \inf\big\{\sup_{y\in A}\|x - y\| : x \in A\big\}$ is called the Chebyshev radius of A. The number $\operatorname{diam}(A) = \sup\{\|x - y\| : x, y \in A\}$ is called the diameter of A. A Banach space X has normal structure (resp. weak normal structure) if

$$r(A) < \operatorname{diam}(A)$$

for every bounded closed (resp. weakly compact) convex subset A of X with $\operatorname{diam}(A) > 0$. X is said to have uniform normal structure (resp. weak uniform normal structure) if

$$\inf\left\{\frac{\operatorname{diam} A}{r(A)}\right\} > 1,$$

where the infimum is taken over all bounded closed (resp. weakly compact) convex subsets A of X with $\operatorname{diam} A > 0$. The weakly convergent sequence coefficient $WCS(X)$ [**?**] of X is the number

$$WCS(X) = \inf\left\{\frac{A(\{x_n\})}{r_a(\{x_n\})}\right\},$$

where the infimum is taken over all sequences $\{x_n\}$ in X which are weakly (not strongly) convergent, $A(\{x_n\}) = \limsup_n\{\|x_i - x_j\| : i, j \geq n\}$ is the asymptotic diameter of $\{x_n\}$, and $r_a(\{x_n\}) = \inf\{\limsup_n \|x_n - y\| : y \in \overline{\text{co}}(\{x_n\})\}$ is the asymptotic radius of $\{x_n\}$.

Some equivalent definitions of the weakly convergent sequence coefficient can be found in [**?**, p. 120] as follows :

$$WCS(X) = \inf\left\{\frac{\lim_{n,m;n\neq m}\|x_n - x_m\|}{\lim_{n\to\infty}\|x_n\|} : x_n \xrightarrow{w} 0,\ \lim_{n,m;n\neq m}\|x_n - x_m\| \text{ and } \lim_{n\to\infty}\|x_n\| \text{ exist}\right\},$$

and

$$WCS(X) = \inf\left\{\lim_{n,m;n\neq m}\|x_n - x_m\| : x_n \xrightarrow{w} 0,\ \|x_n\| = 1 \text{ and } \lim_{n,m;n\neq m}\|x_n - x_m\| \text{ exists}\right\}.$$

It is known that $WCS(X) > 1$ imply X has weak uniform normal structure [**?**].

For a Banach space X, the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ of X, introduced by Clarkson [**?**], is defined by

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x + y\|^2 + \|x - y\|^2}{2\|x\|^2 + 2\|y\|^2} : x, y \in X \text{ not both zero}\right\}.$$

The constant $R(a, X)$, which is a generalized Garcia-Falset coefficient [**?**], is introduced by Dominguez [**?**] : For a given nonnegative real number a,

$$R(a, X) := \sup\{\liminf_n \|x + x_n\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequence $\{x_n\}$ in the unit ball of X such that $\lim_{n,m;n\neq m} \|x_n - x_m\| \leq 1$.

A relationship between the constant $R(1, X)$ and the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ can be found in [?] :

$$R(1, X) \leq \sqrt{2C_{\mathrm{NJ}}(X)}.$$

The following method and results deal with the concept of asymptotic centers. Let E be a nonempty bounded closed subset of X and $\{x_n\}$ a bounded sequence in X. We use $r(E, \{x_n\})$ and $A(E, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in E, respectively, i.e.,

$$r(E, \{x_n\}) = \inf \big\{ \limsup_{n\to\infty} \|x_n - x\| : x \in E \big\},$$

$$A(E, \{x_n\}) = \big\{ x \in E : \limsup_{n\to\infty} \|x_n - x\| = r(E, \{x_n\}) \big\}.$$

It is known that $A(E, \{x_n\})$ is a nonempty weakly compact convex set as E is [?].

Let $\{x_n\}$ and E be as above. Then $\{x_n\}$ is called regular relative to E if $r(E, \{x_n\}) = r(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{x_n\}$ is called asymptotically uniform relative to E if $A(E, \{x_n\}) = A(E, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. Furthermore, $\{x_n\}$ is called regular asymptotically uniform relative to E if $\{x_n\}$ is regular and asymptotically uniform relative to E.

Lemma 2.1 (Goebel [?], Lim [?]). *Let $\{x_n\}$ and E be as above. Then*
(i) there always exists a subsequence of $\{x_n\}$ which is regular relative to E;
(ii) if E is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform relative to E.

A last concept which we need to mention is ultrapowers of Banach spaces. Ultrapowers are proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. We recall some basic facts about ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i\in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I, i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the following facts :

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and
(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space $\Pi_{i\in I} X_i$ equipped with the norm $\|\{x_i\}\| := \sup_{i\in I} \|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \big\{\{x_i\} \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\big\}.$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $\{x_i\}_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from (ii) and the definition of the quotient norm that

$$\|\{x_i\}_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|.$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an ultrapower of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see Aksoy and Khamsi [**?**] or Sims [**?**]).

3. Main results

Definition 3.1. A Banach space X is said to satisfy property (D) if there exists $\lambda \in [0, 1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\} \subset E$ which is regular asymptotically uniform relative to E, and any sequence $\{y_n\} \subset A(E, \{x_n\})$ which is regular asymptotically uniform relative to E we have

$$r(E, \{y_n\}) \leq \lambda r(E, \{x_n\}). \tag{3.1}$$

Theorem 3.2. *Let X be a Banach space satisfying property (D). Then X has weak normal structure.*

Proof. Suppose by contradiction, thus there exists a weakly null sequence $\{x_n\} \subset B_X$ such that $\lim_{n\to\infty} \|x_n - x\| = 1$ for all $x \in C = \overline{\text{co}}(\{x_n\})$ (see [**?**]). By passing through a subsequence, we may assume that $\{x_n\}$ is regular relative to C. We see that $r(C, \{x_n\}) = 1$ and $A(C, \{x_n\}) = C$. Moreover $\{x_n\}$ is asymptotically uniform relative to C. Indeed, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ we have

$$A(C, \{x_{n_k}\}) = \big\{x \in C : \limsup_{k\to\infty} \|x_{n_k} - x\| = r(C, \{x_{n_k}\})\big\} = C.$$

Since $\{x_n\} \subset C = A(C, \{x_n\})$ and X satisfies property (D) with a corresponding $\lambda \in [0, 1)$,

$$r(C, \{x_n\}) \leq \lambda r(C, \{x_n\})$$

which leads to a contradiction. □

The following results will be very useful in order to prove our main theorem.

Theorem 3.3 (Dominguez-Lorenzo [**?**])**.** *Let E be a nonempty weakly compact separable subset of a Banach space X, $T : E \to K(E)$ a nonexpansive mapping,* *and* $\{x_n\}$ *a sequence in* E *such that* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. *Then there exists a subsequence* $\{z_n\}$ *of* $\{x_n\}$ *such that*

$$Tx \cap A \neq \emptyset, \quad \forall x \in A := A(E, \{z_n\}).$$

Theorem 3.4 (Dominguez-Lorenzo [**?**])**.** *Let* E *be a nonempty weakly compact convex separable subset of a Banach space* X. *Assume that* $T : E \to KC(E)$ *is a contraction mapping. If* A *is a closed convex subset of* E *such that* $Tx \cap A \neq \emptyset$ *for all* $x \in A$, *then* T *has a fixed point in* A.

We can now state our main theorem.

Theorem 3.5. *Let* E *be a nonempty weakly compact convex subset of a Banach space* X *which satisfies property (D). Assume that* $T : E \to KC(E)$ *is a nonexpansive mapping. Then* T *has a fixed point.*

Proof. The first part of the proof is similar to the proof of Theorem 4.2 in [**?**]. Therefore, we only sketch this part of the proof. From [**?**] we can assume that E is separable. Fix $z_0 \in E$ and define a contraction $T_n : E \to KC(E)$ by

$$T_n(x) = \frac{1}{n} z_0 + (1 - \frac{1}{n})Tx, \quad x \in E.$$

By Nadler's theorem [**?**], for any $n \in \mathbb{N}$, T_n has a fixed point, say x_n^1. It is easy to prove that $\lim_{n\to\infty} \mathrm{dist}(x_n^1, Tx_n^1) = 0$. By Lemma **??**, we can assume that sequence $\{x_n^1\} \subset E$ is a regular asymptotically uniform relative to E. Denote $A_1 = A(E, \{x_n^1\})$. By Theorem **??** we can assume that $Tx \cap A_1 \neq \emptyset$ for all $x \in A_1$. Fix $z_1 \in A_1$ and define a contraction $T_n : E \to KC(E)$ by

$$T_n(x) = \frac{1}{n} z_1 + (1 - \frac{1}{n})Tx, \quad x \in E.$$

Convexity of A_1 implies $T_n(x) \cap A_1 \neq \emptyset$ for all $x \in A_1$. By Theorem **??**, T_n has a fixed point in A_1, say x_n^2. Consequently, we can get a sequence $\{x_n^2\} \subset A_1$ which is regular asymptotically uniform relative to E and $\lim_{n\to\infty} \mathrm{dist}(x_n^2, Tx_n^2) = 0$. Since X satisfies the property (D) with a corresponding $\lambda \in [0, 1)$, we have

$$r(E, \{x_n^2\}) \leq \lambda r(E, \{x_n^1\}).$$

By induction, we can find a sequence $\{x_n^k\} \subset A_{k-1} = A(E, \{x_n^{k-1}\})$ which is regular asymptotically uniform relative to E,

$$\lim_{n\to\infty} \mathrm{dist}(x_n^k, Tx_n^k) = 0,$$

and

$$r(E, \{x_n^k\}) \leq \lambda r(E, \{x_n^{k-1}\}) \text{ for all } k \in \mathbb{N}.$$

Consequently,

$$r(E, \{x_n^k\}) \leq \lambda r(E, \{x_n^{k-1}\}) \leq ... \leq \lambda^{k-1} r(E, \{x_n^1\}).$$

In view of [**?**, p. 48], we may assume that for each $k \in \mathbb{N}$,

$$\lim_{n,m;n\neq m} \|x_n^k - x_m^k\| \quad \text{exists},$$

and in addition $\|x_n^k - x_m^k\| < \lim\limits_{n,m;n\neq m} \|x_n^k - x_m^k\| + \frac{1}{2^k}$ for all $n, m \in \mathbb{N}$ and $n \neq m$. Let $\{y_n\}$ be the diagonal sequence $\{x_n^n\}$. We claim that $\{y_n\}$ is a Cauchy sequence. For each $n \geq 1$, we have for any positive number m,

$$\begin{aligned}\|y_n - y_{n-1}\| &\leq \|y_n - x_m^{n-1}\| + \|x_m^{n-1} - y_{n-1}\| \\ &= \|y_n - x_m^{n-1}\| + \|x_m^{n-1} - x_{n-1}^{n-1}\| \\ &\leq \|y_n - x_m^{n-1}\| + \lim_{i,j;i\neq j} \|x_i^{n-1} - x_j^{n-1}\| + \frac{1}{2^{n-1}}\end{aligned}$$

Taking upper limit as $m \to \infty$,

$$\begin{aligned}\|y_n - y_{n-1}\| &\leq \limsup_{m\to\infty} \|y_n - x_m^{n-1}\| + \lim_{i,j;i\neq j} \|x_i^{n-1} - x_j^{n-1}\| + \frac{1}{2^{n-1}} \\ &\leq r(E, \{x_n^{n-1}\}) + \limsup_i \|x_i^{n-1} - y_n\| + \limsup_j \|x_j^{n-1} - y_n\| + \frac{1}{2^{n-1}} \\ &\leq 3r(E, \{x_n^{n-1}\}) + \frac{1}{2^{n-1}} \\ &\leq 3\lambda^{n-2} r(E, \{x_n^1\}) + \frac{1}{2^{n-1}}\end{aligned}$$

Since $\lambda < 1$, we conclude that there exists $y \in E$ such that y_n converges to y. Consequently,

$$\operatorname{dist}(y, Ty) \leq \|y - y_n\| + \operatorname{dist}(y_n, Ty_n) + H(Ty_n, Ty) \to 0 \text{ as } n \to \infty$$

Hence y is a fixed point of T. □

Theorem 3.6. *Let E be a nonempty weakly compact convex subset of a Banach space X with*

$$C_{NJ}(X) < 1 + \frac{WCS(X)^2}{4}.$$

Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point.

Proof. We will prove that X satisfies property (D). Since $C_{\mathrm{NJ}}(X) < 1 + \frac{WCS(X)^2}{4}$, we choose $\lambda = \frac{2\sqrt{C_{\mathrm{NJ}}(X)-1}}{WCS(X)} < 1$. Let $\{x_n\} \subset E$ and $\{y_n\} \subset A(E, \{x_n\})$ be regular asymptotically uniform sequences relative to E. We will show that (**??**) is satisfied. Let $\varepsilon > 0$. Choosing a subsequence, if necessary, we can assume that $\{y_n\}$ converges weakly to $y \in E$,

$$\lim_{k,j;k\neq j} \|y_k - y_j\| = l \quad \text{for some } l \in \mathbb{R},$$

and

$$\big|\|y_k - y_j\| - l\big| < \varepsilon \quad \text{for all } k \neq j. \tag{3.2}$$

Let $r = r(E, \{x_n\})$ and fix $k \neq j$. Since $y_k, y_j \in A(E, \{x_n\})$ and using the convexity of $A(E, \{x_n\})$, we can assume, passing through a subsequence, that

$$\|x_n - y_k\| < r + \varepsilon, \ \|x_n - y_j\| < r + \varepsilon, \tag{3.3}$$

and

$$\Big\|x_n - \frac{y_k + y_j}{2}\Big\| > r - \varepsilon \quad \text{for all large } n. \tag{3.4}$$

From the definition of $C_{\rm NJ}(X)$, by (**??**), (**??**), and (**??**) we have for n large enough

$$\begin{aligned} C_{\rm NJ}(X) &\geq \frac{\|2x_n - (y_k + y_j)\|^2 + \|y_k - y_j\|^2}{2\|x_n - y_k\|^2 + 2\|x_n - y_j\|^2} \\ &\geq \frac{4(r-\varepsilon)^2 + (l-\varepsilon)^2}{4(r+\varepsilon)^2} \end{aligned}$$

Since ε is arbitrary, it follows that

$$C_{\rm NJ}(X) \geq \frac{4r^2 + l^2}{4r^2}$$

Since

$$WCS(X) = \inf\Big\{\frac{\lim_{j,k;j\neq k}\|u_j - u_k\|}{\limsup_j \|u_j\|} : u_j \xrightarrow{w} 0, \lim_{j,k;j\neq k}\|u_j - u_k\| \text{ exists}\Big\},$$

we can deduce that

$$\begin{aligned} C_{\rm NJ}(X) &\geq 1 + \frac{WCS(X)^2(\limsup_n \|y_n - y\|)^2}{4r^2} \\ &\geq 1 + \frac{WCS(X)^2 r(E, \{y_n\})^2}{4r^2}. \end{aligned}$$

Consequently,

$$r(E, \{y_n\}) \leq \frac{2\sqrt{C_{\rm NJ}(X) - 1}}{WCS(X)} r = \lambda r(E, \{x_n\})$$

as desired. □

In order to prove our next result, we need the following theorem which states a relationship between the weakly convergent sequence coefficient and the Jordan-von Neumann constant of a Banach space X.

Theorem 3.7. *For a Banach space X,*

$$[WCS(X)]^2 \geq \frac{2C_{\rm NJ}(X) + 1}{2[C_{\rm NJ}(X)]^2}.$$

Proof. Since $C_{\mathrm{NJ}}(X) \leq 2$ and the result is obvious if $C_{\mathrm{NJ}}(X) = 2$, we can assume that $C_{\mathrm{NJ}}(X) < 2$. It is known that $C_{\mathrm{NJ}}(X) < 2$, or equivalently X is uniformly nonsquare, implies X and X^* are reflexive. Put $\alpha = \sqrt{2C_{\mathrm{NJ}}(X)}$. Let $\{x_n\}$ be a normalized weakly null sequence in X. Put $d = \lim_{n,m;n\neq m} \|x_n - x_m\|$. We know that $WCS(X) \leq d$. Consider a sequence $\{f_n\}$ of norm one functionals for which $f_n(x_n) = 1$. Since X^* is reflexive we can assume that $\{f_n\}$ converges weakly to some f in X^*. Let ε be an arbitrary positive number and choose $K \in \mathbb{N}$ large enough so that $|f(x_n)| < \varepsilon$ and $d - \varepsilon \leq \|x_n - x_m\| \leq d + \varepsilon$ for any $m \neq n;\ m, n \geq K$. Then we have

$$\lim_n (f_n - f)(x_K) = 0 \quad \text{and} \quad \lim_n f_K(x_n) = 0.$$

Since $\lim\limits_{n,m;n\neq m} \|\frac{x_n - x_m}{d+\varepsilon}\| < 1$ and $\|\frac{x_K}{d+\varepsilon}\| \leq 1$, we have, by definition of $R(1, X)$,

$$\limsup_n \|x_n + x_K\| \leq (d+\varepsilon)R(1, X) \leq (d+\varepsilon)\sqrt{2C_{\mathrm{NJ}}(X)} = (d+\varepsilon)\alpha.$$

We construct elements of $\widetilde{X}$ and $\widetilde{X^*}$.

$$\tilde{x} = \left\{\frac{x_n - x_K}{d+\varepsilon}\right\}_{\mathcal{U}} \quad \text{and} \quad \tilde{y} = \left\{\frac{x_n + x_K}{(d+\varepsilon)\alpha}\right\}_{\mathcal{U}} \quad ,$$

$$\tilde{f} = \{f_n\}_{\mathcal{U}} \quad \text{and} \quad \tilde{g} = \dot{f_K}.$$

Here $\dot{x}$ denotes an equivalence class of a sequence$\{x_n\}$ such that $x_n \equiv x$ for all $n \in \mathbb{N}$. It is easy to see that $\tilde{x}, \tilde{y} \in B_{\widetilde{X}}$ and $\tilde{f}, \tilde{g} \in B_{\widetilde{X^*}}$. Moreover,

$$\tilde{f}(\{x_n\}_{\mathcal{U}}) = 1 \quad \text{and} \quad |\tilde{f}(\dot{x_K})| = |\dot{f}(\dot{x_K})| < \varepsilon.$$

On the other hand,

$$\tilde{g}(\{x_n\}_{\mathcal{U}}) = 0 \quad \text{and} \quad \tilde{g}(\dot{x_K}) = 1.$$

Let consider

$$\begin{aligned}
\|\tilde{f} - \tilde{g}\| &\geq (\tilde{f} - \tilde{g})(\tilde{x}) = \tilde{f}(\tilde{x}) - \tilde{g}(\tilde{x}) \\
&= \frac{1}{d+\varepsilon}\big(\tilde{f}(\{x_n\}_{\mathcal{U}}) - \tilde{f}(\dot{x_K}) - [\tilde{g}(\{x_n\}_{\mathcal{U}}) - \tilde{g}(\dot{x_K})]\big) \\
&\geq \frac{1}{d+\varepsilon}\big(1 - \varepsilon - 0 + 1\big) = \frac{2-\varepsilon}{d+\varepsilon}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\tilde{f}+\tilde{g}\| &\geq (\tilde{f}+\tilde{g})(\tilde{y}) = \tilde{f}(\tilde{y})+\tilde{g}(\tilde{y}) \\ &= \frac{1}{(d+\varepsilon)\alpha}\big(\tilde{f}(\{x_n\}_{\mathcal{U}})+\tilde{f}(\dot{x_K})+\tilde{g}(\{x_n\}_{\mathcal{U}})+\tilde{g}(\dot{x_K})\big) \\ &\geq \frac{1}{(d+\varepsilon)\alpha}\big(1-\varepsilon+0+1\big) = \frac{2-\varepsilon}{(d+\varepsilon)\alpha}.\end{aligned}$$

Thus we have

$$\begin{aligned}C_{\mathrm{NJ}}(\widetilde{X^*}) &\geq \frac{\|\tilde{f}+\tilde{g}\|^2+\|\tilde{f}-\tilde{g}\|^2}{2\|\tilde{f}\|^2+2\|\tilde{g}\|^2} \\ &\geq \frac{(\frac{2-\varepsilon}{d+\varepsilon})^2+(\frac{2-\varepsilon}{(d+\varepsilon)\alpha})^2}{4} \\ &= (\frac{1}{d+\varepsilon})^2(\frac{(2-\varepsilon)^2}{4}+\frac{(2-\varepsilon)^2}{4\alpha^2}).\end{aligned}$$

Since ε is arbitrary, $\alpha = \sqrt{2C_{\mathrm{NJ}}(X)}$, and the Jordan-von Neumann constants of $X^*, X, \widetilde{X}$ and $\widetilde{X}^*$ are all equal, we obtain

$$C_{\mathrm{NJ}}(X) \geq (\frac{1}{d^2})(1+\frac{1}{2C_{\mathrm{NJ}}(X)}).$$

Thus

$$[WCS(X)]^2 \geq \frac{2C_{\mathrm{NJ}}(X)+1}{2[C_{\mathrm{NJ}}(X)]^2}.$$

□

Using Theorem **??**, we obtain the following corollary.

Corollary 3.8. [**?**, Theorem 3.16] *Let X be a Banach space. If $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$, then X and X^* has uniform normal structure.*

Proof. Let $\widetilde{X}$ be a Banach space ultrapower of X over an ultrafilter. Since $C_{\mathrm{NJ}}(\widetilde{X}) = C_{\mathrm{NJ}}(X)$, Theorem **??** can be applied to $\widetilde{X}$. The inequality in Theorem **??** implies $WCS(\widetilde{X}) > 1$ if $C_{\mathrm{NJ}}(\widetilde{X}) < \frac{1+\sqrt{3}}{2}$. Since $WCS(\widetilde{X}) > 1$ implies $\widetilde{X}$ has weak normal structure [**?**] and since $\widetilde{X}$ is reflexive, it must be the case that $\widetilde{X}$ has normal structure. By [**?**, Theorem 5.2], X has uniform normal structure as desired. □

Using the inequality appearing in Theorem **??**, and numerical calculus it is not difficult to check that $C_{\mathrm{NJ}}(X) < 1 + \frac{WCS(X)^2}{4}$ if $C_{\mathrm{NJ}}(X) < c_0 = 1.273...$ Thus we can state :

Corollary 3.9. *Let E be a nonempty bounded closed convex subset of a Banach space X with*

$$C_{NJ}(X) < c_0 = 1.273...$$

Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point.

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