A UNIVERSAL INFINITE-DIMENSIONAL MODULUS FOR NORMED SPACES AND APPLICATIONS

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Abstract. We define an infinite-dimensional modulus which can be simultaneously considered as a measure of nearly uniform convexity and nearly uniform smoothness. We compute this modulus in some classical Banach spaces and we show some basic properties of this modulus and some applications to prove that a Banach space satisfies the w-fixed point property for non-expansive mappings.

1. Introduction

In 1995 C. Benítez, K. Przeslawski and D. Yost [**?**] defined a two-dimensional modulus for normed spaces. Given a normed space X, one observes that for any $x, y \in X$ with $\|y\| < 1 < \|x\|$, there is a unique $z = z(x, y)$ in the line segment $[x, y]$ with $\|z\| = 1$. They define $\xi_X : [0, 1) \to \mathbb{R}$ by

$$\xi_X(\beta) = \sup\left\{\frac{\|x - z(x,y)\|}{\|x\| - 1} : \|y\| \leq \beta < 1 < \|x\|\right\}$$

They called ξ modulus of squareness because its extreme values characterize nearly-squareness (we recall that X is nearly square if for all $\epsilon > 0$ there exists Y subspace of X with dim $Y = 2$ such that $d(Y, \ell_1(2)) < 1 + \epsilon$, where $d(E, F)$ is the Banach-Mazur distance between two normed spaces E and F). The behaviour of the modulus of squareness is strongly connected with the geometry of space. In particular, the modulus of squareness tells us whether or not a space is uniformly smooth, uniformly convex, uniformly non-square or an inner product space. Furthermore, this modulus can also be used to obtain

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uniform normal structure.
The main properties of the modulus ξ are collected in the following theorem:

Theorem 1.1. [?] *Let X be any normed space and ξ its modulus of squareness. Then:*

(1) $\xi(\beta) = \sup\{\xi_M(\beta) : M \subset X, \dim M = 2\}$
(2) *ξ is strictly increasing and convex*
(3) *$\xi(\beta) < \xi_1(\beta)$ for each $\beta \in (0,1)$, unless X is nearly square, in that case $\xi(\beta) = \xi_1(\beta)$ for all $\beta \in (0,1)$, where*
$$\xi_1(\beta) = \frac{1+\beta}{1-\beta}.$$
In particular, if X is non-reflexive, $\xi(\beta) = \xi_1(\beta)$ for all $\beta \in (0,1)$
(4) *$\xi' \leq \xi_1'$ almost everywhere on $(0,1)$*
(5) *$\xi(\beta) > \xi_2(\beta)$ for each $\beta \in (0,1)$, unless X is an inner product space, in that case $\xi(\beta) = \xi_2(\beta)$ for all $\beta \in (0,1)$, where*
$$\xi_2(\beta) = \frac{1}{\sqrt{1-\beta^2}}$$
(6) *Let X and Y be two isomorphic normed spaces whose Banach-Mazur distance is less than $1+\delta^2$ where $\delta \leq 1$. Then*
$$|\xi_X(\beta) - \xi_Y(\beta)| \leq \frac{2(\delta+\delta^2)}{(1-\beta)^2} \text{ for all } \beta \in (0,1)$$
(7) *X is uniformly convex if and only if $\lim_{\beta\to 1}(1-\beta)\xi(\beta) = 0$*
(8) *X is uniformly smooth if and only if $\xi'(0) = 0$*
(9) *The modulus of squareness of X^* at β is $\xi_{X^*}(\beta) = 1/\xi^{-1}(1/\beta)$*
(10) *If $\xi(\beta) < 1/(1-\beta)$ for some β, then X has uniform normal structure.*

The modulus of squareness has an advantage with respect to other previously defined moduli: it is simultaneously suitable for the uniform convexity and the uniform smoothness of the space. This modulus, just as uniform convexity and uniform smoothness, has a finite-dimensional character, that is, it only depends on the finite-dimensional subspaces of the space. It is well known that nearly uniform convexity and nearly uniform smoothness are natural generalizations of uniform convexity and uniform smoothness (respectively) and they have infinite dimensional character. These notions have proved to be very useful in Fixed Point Theory and Geometric Theory of Banach spaces. In this paper we define a new infinite-dimensional modulus ζ which can be simultaneously considered as a measure of nearly uniform convexity and nearly uniform smoothness. We study in depth this new modulus, computing its value in ℓ_p-spaces and c_0, proving some basic properties and its connections with other important geometric properties of Banach spaces yielding to some fixed point results.

2. Preliminaries

In this section we introduce some known results related to the existence of fixed points for nonexpansive mappings which will be used through this paper. For more details the reader may consult, for instance [**?**] and [**?**].

Throughout this paper X will be a Banach space. We say that X has the weak fixed point property (w-FPP) if every nonexpansive mapping T defined from a weakly compact convex subset C of X into C has a fixed point. We say that X has weak normal structure (w-NS) if every weakly compact convex subset of X with more than one member is not diametral.

Theorem 2.1. [**?**]*If X has w-NS then X has the w-FPP.*

Associated to the weak normal structure of a Banach space, Bynum [**?**] defined the weakly convergent sequence coefficient. We shall use an equivalent definition [**?**]

Let X be a Banach space without Schur property. The weakly convergent sequence coefficient of X is defined by

$$WCS(X) = \inf\left\{\frac{\lim_{n,m;n\neq m}\|x_n - x_m\|}{\lim_n \|x_n\|}\right\}$$

where the infimum is taken over all weakly null sequences $\{x_n\}$ such that both limits exist and $\lim_n \|x_n\| \neq 0$.

It must be noted that every bounded sequence contained in a metric space has a subsequence $\{x_n\}$ such that $\lim_{n,m;n\neq m} d(x_n, x_m)$ exists (see [**?**] Theorem III.1.5).

Theorem 2.2. [**?**] *If $WCS(X) > 1$ then X has w-NS.*

We say that X has weak uniform normal structure (w-UNS) if $WCS(X) > 1$.

In the following theorem we recall the value of $WCS(X)$ in some particular Banach spaces. Previously we recall the definition of Bynum spaces.

Let $p \in [1, +\infty)$, $q \in [1, +\infty]$. The Bynum spaces $\ell_{p,q}$ are defined as $\ell_{p,q} = (\ell_p, \|\cdot\|_{p,q})$ where

$$\|x\|_{p,q} = (\|x^+\|_p^q + \|x^-\|_p^q)^{1/q} \quad \text{if } q \in [1, +\infty)$$

$$\|x\|_{p,\infty} = \max\{\|x^+\|_p, \|x^-\|_p\}$$

x^+, x^- denote the positive and the negative part of x respectively and $\|x\|_p$ denotes the norm of x in ℓ_p.

Theorem 2.3. [**?**][**?**]

(1) *For every real number* $p \geq 1$*, we have* $WCS(\ell_p) = 2^{1/p}$
(2) $WCS(c_0) = 1$
(3) *For* $p > 1$*,* $q \geq 1$*, we have* $WCS(\ell_{p,q}) = \min\left\{2^{1/p}, 2^{1/q}\right\}$

Now we are going to consider some geometric properties connected with normal structure.

The space X is said to be uniformly convex (UC) if for each $\epsilon \in (0,2]$ there exists $\delta > 0$ such that for $x, y \in X$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$ then $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$.

The space X is said to be uniformly smooth (US) if

$$\lim_{t \to 0^+} \frac{\rho_X(t)}{t} = 0$$

where

$$\rho_X(t) = \sup\left\{\frac{\|x + ty\| + \|x - ty\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq 1\right\}$$

Next we recall the definitions of nearly uniform convexity and nearly uniform smoothness which are infinite-dimensional generalizations of the notions of uniform convexity and uniform smoothness.

The space X is said to be nearly uniformly convex (NUC) if X is reflexive and $\Delta_X(\epsilon) > 0$ for each $\epsilon > 0$, i.e., $\Delta_0(X) = 0$, where

$$\Delta_X(\epsilon) = \inf\{1 - \|x\| : \{x_n\} \subset B_X, x_n \rightharpoonup x, \liminf_n \|x_n - x\| \geq \epsilon\}$$

$$\Delta_0(X) = \sup\{\epsilon > 0 : \Delta_X(\epsilon) = 0\}$$

The space X is said to be nearly uniformly smooth (NUS) if X is reflexive and for any $\epsilon > 0$ there exists $\eta > 0$ such that for any $t \in (0, \eta)$ and any weakly null sequence $\{x_n\} \subset B_X$ there exists $k > 1$ such that $\|x_1 + tx_k\| \leq 1 + \epsilon t$.

Notice that every NUC space has normal structure ([**?**] Remark VI.4.7). The situation for NUS spaces is different. Indeed, $\ell_{p,1}$ is NUC, so its dual $\ell_{q,\infty}$ is NUS but this space fails to have normal structure ([**?**] Example VI.2). However NUS spaces have the fixed point property [**?**].

Up to now we have deduced the existence of fixed points for nonexpansive mappings as a consequence of normal structure. Now we recall other geometric properties of Banach spaces which give fixed point results without normal structure.

In 1991 J. García Falset [**?**] defined the following geometric coefficient

$$R(X) = \sup\left\{\liminf_{n \to \infty} \|x_n + x\| : \{x_n\}, x \in B_X, x_n \rightharpoonup 0\right\}$$

and, later, he proved the following theorem

Theorem 2.4. [**?**] *A Banach space* X *has the w-FPP if* $R(X) < 2$.

3. A universal infinite-dimensional modulus

The first problem appearing when we try to define a modulus suitable for NUC and NUS is the non existence of a relationship between these concepts for a fixed space. Indeed, if X is at the worst situation for uniform convexity, i.e. $\varepsilon_0(X) = 2$, then $\xi'_X(0) = 2$ and so X is not uniformly smooth (see [**?**] Theorem 2.4). However there is no a similar behaviour for nearly uniform convexity and nearly uniform smoothness as the following example shows.

Example 3.1. Let X be the Bynum space $\ell_{2,\infty}$. Since $\ell_{2,1}$ is NUC ([**?**] Example V.1), its dual $\ell_{2,\infty}$ is NUS. We recall that for any bounded set $A \subset X$, the Hausdorff measure of noncompactness is defined by $\chi(A) = \inf\{\epsilon >=: A$ has a finite $\epsilon-$net$\}$. The characteristic of noncompact convexity of X is defined by $\varepsilon_\chi(X) = \sup\{\epsilon > 0 : \Delta_\chi(\epsilon) = 0\}$ where $\Delta_\chi(\epsilon) = \inf\{1 - d(0, A) : A \subset B_X$ is convex , $\chi(A) > \varepsilon\}$. We are going to show that $\varepsilon_\chi(X) = 1$, that is, the maximum value for ε_χ. Let A be the convex hull of the sequence $x_n = e_1 - e_n$, which is w-convergent to e_1. Since $\lim_n \|x_n - e_1\|$ exists and $\ell_{2,\infty}$ satisfies the non strict Opial condition [**?**], using [**?**] Lemma 1.1 we have

$$\chi(\{x_n\}) = \lim_n \|x_n - e_1\| = \lim_n \|e_n\| = 1.$$

Since the measure χ satisfies the invariance under passage to the convex hull [**?**], we have $\chi(A) = 1$ and $\varepsilon_\chi(X) = 1$.

Definition 3.2. *Let X be a Banach space. For each $\beta \in (0, 1)$, the* universal infinite-dimensional modulus *is defined by*

$$\zeta_X(\beta) = \sup\left\{\liminf_{n\to\infty} \frac{\|x_n - y\|}{1 - \|x\|}\right\}$$

where the supremum is taken over all sequences $\{x_n\} \subset B_\beta$ such that $x_n \rightharpoonup x \neq 0, \liminf_n \|x_n - x\| \leq \beta$ and $y = \frac{x}{\|x\|}$, where B_β denotes the closed ball centered at 0 with radius β.

Remark: Since the norm $\|\cdot\|$ is w-sequentially lower semicontinuous (w-slsc), for each $\beta \in (0, 1)$ we have

$$1 \leq \zeta_X(\beta) \leq \frac{1}{1-\beta}.$$

In the following theorem we will show that both extreme values are attained in some spaces.

Theorem 3.3. (1) *If X satisfies Schur property, that is, every weakly convergent sequence is norm convergent, then $\zeta_X(\beta) = 1$ for any $\beta \in (0, 1)$.*
(2) $\zeta_{\ell_\infty}(\beta) = \frac{1}{1-\beta}$ *for each $\beta \in (0, 1)$.*

(3) $\zeta_{\ell_2}(\beta) = \xi_{\ell_2}(\beta) = \frac{1}{\sqrt{1-\beta^2}}$ *for each* $\beta \in (0,1)$.
(4) *If* $1 < p < \infty$*, then* $\zeta_{\ell_p}(\beta) = \frac{((1-\beta^q)^{p-1}+\beta^p)^{1/p}}{(1-\beta^q)^{1/q}}$ *for each* $\beta \in (0,1)$.
(5) $\zeta_{c_0}(\beta) = \max\left(1, \frac{\beta}{1-\beta}\right)$ *for each* $\beta \in (0,1)$.

Proof.

(1) Obvious
(2) Denote $1 = \sum_{n=1}^{\infty} e_n$ and consider the sequence $x_n = \beta(1 - e_n) \in B_\beta$ which converges weakly to $\beta 1 =: x$ with $\liminf_n \|x_n - x\| \le \beta$. Furthermore, $\|x_n - y\| = 1$ for every n. Easily we deduce the claimed value of $\zeta_{\ell_\infty}(\beta)$.
(3) The following result can be derived from [**?**]: Let $\{x_n\}$ be a w-null sequence in ℓ_p, $1 \le p < +\infty$. Then

$$\liminf_{n\to\infty} \|x_n - x\|^p = \liminf_{n\to\infty} \|x_n\|^p + \|x\|^p. \tag{3.1}$$

for every $x \in \ell_p$.

Let $\{x_n\}$ be a sequence in $B_\beta \subset \ell_2$ which is weakly convergent to x (which implies $\liminf_n \|x_n - x\| \le \beta$ because ℓ_2 satisfies the Opial property). We denote $A = \|x\|$ and, using (**??**), we have

$$\beta^2 \ge \liminf_n \|x_n\|^2 = \liminf_n \|x_n - x\|^2 + A^2$$

and

$$\liminf_n \frac{\|x_n - y\|^2}{(1-\|x\|)^2} = \liminf_n \frac{\|x_n - x\|^2}{(1-\|x\|)^2} + 1 \le \frac{\beta^2 - A^2}{(1-A)^2} + 1.$$

Elementary calculus proves that the function $f(A) = (\beta^2 - A^2)/(1-A)^2$ attains its maximum in $[0,\beta]$ at the point $A = \beta^2$. By substitution of this value we obtain

$$\zeta_{\ell_2}(\beta) \le \frac{1}{\sqrt{1-\beta^2}}$$

and this upper bound is attained for $x_n = \beta^2 e_1 + \beta\sqrt{1-\beta^2}e_n$.
(4) It is analogous to the case $p = 2$.
(5) Let $\{x_n\} \subset B_\beta$ be a sequence which converges weakly to x and $\liminf_n \|x_n - x\| \le \beta$. Taking a subsequence, if necessary, we can assume

$$\operatorname{supp}(x_n - x) \cap \operatorname{supp}(x - y) = \emptyset$$

where for every $x = (x^k)$ we denote $\operatorname{supp} x = \{k : x^k \ne 0\}$. We have

$$\liminf_n \|x_n - y\| = \liminf_n \|x_n - x + x - y\| =$$
$$= \max(\liminf_n \|x_n - x\|, \|x - y\|) \le \max(\beta, 1 - \|x\|).$$

Thus

$$\zeta_{c_o}(\beta) \le \frac{\max(\beta, 1-\|x\|)}{1-\|x\|} \le \max\left(\frac{\beta}{1-\beta}, 1\right)$$

and this value is attained at $x_n = \beta(e_1 + e_n)$.

Now we study some basic properties of the new modulus. The following is obvious:

Proposition 3.4. *Let X be a Banach space. The function $\zeta_X(\cdot)$ is increasing.*

In general $\zeta_X(\cdot)$ is not strictly increasing, consider for example a Banach space X with Schur property or $X = c_0$ (see Theorem **??**).

On the other hand this modulus is convex what is noteworthy having in mind that the modulus of convexity need not be a convex function.

Proposition 3.5. *For every Banach space X, $\zeta_X(\cdot)$ is a convex function.*

Proof. Consider $0 < \beta_1 < \beta_2 < 1$, $0 < t < 1$. We have to prove

$$\zeta_X(t\beta_1 + (1-t)\beta_2) \le t\zeta_X(\beta_1) + (1-t)\zeta_X(\beta_2).$$

Let $\{x_n\} \subset B_{t\beta_1+(1-t)\beta_2}$ such that $x_n \rightharpoonup x$, $\liminf_n \|x_n - x\| \le t\beta_1 + (1-t)\beta_2$ and $y = x/\|x\|$. It is enough to find two sequences $\{x_n^1\} \subset B_{\beta_1}$, $\{x_n^2\} \subset B_{\beta_2}$ such that

$$\liminf_{n\to\infty} \frac{\|x_n - y\|}{1-\|x\|} \le t \liminf_n \frac{\|x_n^1 - y^1\|}{1-\|x^1\|} + (1-t)\liminf_n \frac{\|x_n^2 - y^2\|}{1-\|x^2\|}.$$

Consider for $i = 1, 2$

$$x_n^i = \frac{\beta_i}{t\beta_1 + (1-t)\beta_2} x_n \rightharpoonup \frac{\beta_i}{t\beta_1 + (1-t)\beta_2} x = x^i \qquad y^i = \frac{x^i}{\|x^i\|} = \frac{x}{\|x\|} = y.$$

Using the convexity of the norm, we have

$$\frac{\|x_n - y\|}{1-\|x\|} \le \frac{t\|x_n^1 - y^1\| + (1-t)\|x_n^2 - y^2\|}{t(1-\|x^1\|) + (1-t)(1-\|x^2\|)}.$$

On the other hand if we consider

$$z_n = \lambda x^2 + (1-\lambda)x_n^2 = \mu y + (1-\mu)x_n^1$$

with

$$\lambda = \frac{1-\frac{\beta_1}{\beta_2}}{1-\|x^1\|} \in (0,1) \qquad \text{and} \qquad \mu = \frac{1-\frac{\beta_1}{\beta_2}}{1-\|x^1\|}\|x^2\| \in (0,1)$$

we have

$$\|z_n - y\| = (1-\mu)\|x_n^1 - y\|$$
$$x^2 = (1-\mu)x^1 + \mu y \Rightarrow 1 - \|x^2\| = (1-\mu)(1-\|x^1\|)$$

Furthermore, using that $\|\cdot\|$ is convex and w-slsc we obtain

$$\liminf_n \|z_n - y\| \le \lambda \|x^2 - y\| + (1-\lambda)\liminf_n \|x_n^2 - y\| \le \liminf_n \|x_n^2 - y\|$$

and we deduce

$$\liminf_n \frac{\|x_n^1 - y\|}{1-\|x^1\|} = \liminf_n \frac{\|z_n - y\|}{1-\|x^2\|} \le \liminf_n \frac{\|x_n^2 - y\|}{1-\|x^2\|}.$$

Thus, using [**?**] Lemma 3.1 we have the desired inequality.

Proposition 3.6. *The function* $\zeta_X(\cdot)$ *is continuous in* $(0,1)$.

Proof. We will prove continuity from the left. Continuity from the right can be proved in the same way. Let $\epsilon > 0$ be arbitrary, choose γ satisfying $1 < \gamma < \frac{\epsilon(1-\beta)^2}{2} + 1$ and $\beta' = \beta/\gamma$. Let $\{x_n\} \subset B_\beta$ be a sequence which converges weakly to x and $\liminf_n \|x_n - x\| \le \beta$. The sequence $\{x_n/\gamma\} \subset B_{\beta'}$ converges weakly to x/γ with $\liminf_n \|x_n/\gamma - x/\gamma\| \le \beta'$. For $y = x/\|x\|$ we have

$$\zeta_X(\beta') \ge \liminf_n \frac{\left\|\frac{x_n}{\gamma} - y\right\|}{1 - \left\|\frac{x}{\gamma}\right\|} = \liminf_n \frac{\|x_n - \gamma y\|}{\gamma - \|x\|} \ge$$

$$\ge \liminf_n \frac{\|x_n - y\|}{\gamma - \|x\|} - \frac{(\gamma-1)\|y\|}{\gamma - \|x\|} \ge$$

$$\ge \liminf_{n\to\infty} \frac{\|x_n - y\|}{1-\|x\|} - \|x_n - y\|\left(\frac{1}{1-\|x\|} - \frac{1}{\gamma - \|x\|}\right) - \frac{\gamma - 1}{1-\beta} \ge$$

$$\ge \liminf_{n\to\infty} \frac{\|x_n - y\|}{1-\|x\|} - \frac{(\gamma-1)(1+\beta) + (\gamma-1)(1-\beta)}{(1-\beta)^2} =$$

$$= \liminf_{n\to\infty} \frac{\|x_n - y\|}{1-\|x\|} - \frac{2(\gamma-1)}{(1-\beta)^2} \ge \liminf_{n\to\infty} \frac{\|x_n - y\|}{1-\|x\|} - \epsilon.$$

Taking supremum we obtain $\zeta_X(\beta') \ge \zeta_X(\beta) - \epsilon$.

Next we obtain the following relationship between modulus ζ and modulus ξ. As a consequence, the new modulus is a refinement of the previous one.

Theorem 3.7. *For any normed space* X *and any* $\beta \in (0,1)$, $\zeta_X(\beta) \le \xi_X(\beta)$.

Proof. Let $\{x_n\}$ be a sequence in B_β which converges weakly to $x \ne 0$ with $\liminf_n \|x_n - x\| \le \beta$. Choose $\gamma > 1$ arbitrary and $z_n \in [x_n, \gamma y]$ such that $\|z_n\| = 1$. There exists $\lambda_n \in (0,1)$ such that $z_n = \lambda_n \gamma y + (1-\lambda_n)x_n$. Thus $\|\lambda_n \gamma y + (1-\lambda_n)x_n\| = 1$. We can assume, without loss of generality, that $\{\lambda_n\}$ is convergent to some λ. We obtain $\|\lambda\gamma y + (1-\lambda)x\| \le 1$, which implies

$$\|x\| \left| \frac{\lambda\gamma}{\|x\|} + 1 - \lambda \right| \le 1$$

and we deduce

$$\lambda(\gamma - \|x\|) \leq 1 - \|x\|$$

so we have

$$1 - \lambda \geq \frac{\gamma - 1}{\gamma - \|x\|}$$

Choose $c < 1$ arbitrary. For n large enough ($n > n_0(c, \gamma)$) we have

$$1 - \lambda_n \geq c \frac{\gamma - 1}{\gamma - \|x\|}$$

Thus

$$\xi_X(\beta) \geq \frac{\|\gamma y - z_n\|}{\gamma - 1} = \frac{(1 - \lambda_n)\|\gamma y - x_n\|}{\gamma - 1} \geq c \frac{\|\gamma y - x_n\|}{\gamma - \|x\|}$$

Hence

$$\xi_X(\beta) \geq c \liminf_n \frac{\|\gamma y - x_n\|}{\gamma - \|x\|} = c \liminf_n \frac{\left\|y - \frac{x_n}{\gamma}\right\|}{1 - \frac{\|x\|}{\gamma}}$$

Taking supremum we obtain $\xi_X(\beta) \geq c\,\zeta_X(\beta/\gamma)$. Since $\zeta_X(\cdot)$ is continuous and c is arbitrary, letting $\gamma \to 1^+$, we obtain $\xi_X(\beta) \geq \zeta_X(\beta)$.

Theorem 3.8. *Let X and Y be two isomorphic Banach spaces whose Banach-Mazur distance is less than $1 + \delta$, where $\delta \leq 1$. Then, for all $\in (0, 1)$,*

$$|\zeta_X(\beta) - \zeta_Y(\beta)| \leq \frac{\delta^2 + 2\delta}{(1 - \beta)^2}$$

Proof. Our hypothesis implies that we may regard X and Y as the same vector space equipped with two equivalent norms, $\|\cdot\|$ and $|||\cdot|||$ respectively, such that

$$\|x\| \leq |||x||| \leq (1 + \delta)\|x\|$$

for every $x \in X$. Let $\{x_n\} \subset B_X(0, \beta)$ be a sequence such that $x_n \rightharpoonup x$ with $\liminf_n \|x_n - x\| \leq \beta$ and $y = x/\|x\|$. Consider the sequence $x'_n = \frac{1}{1+\delta} x_n \in B_Y(0, \beta)$ which converges weakly to $x' = \frac{1}{1+\delta} x$ with $\liminf_n |||x'_n - x'||| \leq \beta$ and $y' = \frac{x'}{|||x'|||} = \frac{x}{|||x|||}$. We have

$$\|x_n - y\| - |||x'_n - y'||| \leq |||x_n - y||| - |||x'_n - y'||| \leq |||x_n - y - x'_n + y'||| \leq$$

$$\leq |||x_n - x'_n||| + |||y' - y||| = |||x_n||| \left|1 - \frac{1}{1 + \delta}\right| + |||x||| \left|\frac{1}{|||x|||} - \frac{1}{\|x\|}\right| \leq$$

$$\leq |||x_n||| \frac{\delta}{1 + \delta} + |||x||| \delta \frac{1}{\|x\|} \leq \beta\delta + (1 + \delta)\delta = \delta^2 + \delta(1 + \beta).$$

On the other hand we have

$$1 - |||x'||| \leq 1 - \frac{1}{1 + \delta}\|x\| = 1 - \|x\| + \frac{\delta}{1 + \delta}\|x\| \leq 1 - \|x\| + \frac{\beta\delta}{1 + \delta}.$$

Thus

$$\frac{\|x_n-y\|}{1-\|x\|}-\frac{|||x_n'-y'|||}{1-|||x'|||}\le\frac{\|x_n-y\|}{1-\|x\|}-\frac{\|x_n-y\|-[\delta^2+\delta(1+\beta)]}{1-\|x\|+\frac{\beta\delta}{1+\delta}}=$$

$$=\frac{\|x_n-y\|\frac{\beta\delta}{1+\delta}+[\delta^2+\delta(1+\beta)](1-\|x\|)}{(1-\|x\|)(1-\|x\|+\frac{\beta\delta}{1+\delta})}\le\frac{\|x_n-y\|}{1-\|x\|}\frac{\frac{\beta\delta}{1+\delta}}{1-\|x\|}+\frac{\delta^2+\delta(1+\beta)}{1-\|x\|}\le$$

$$\le\frac{1+\beta}{1-\beta}\frac{\frac{\beta\delta}{1+\delta}}{1-\beta}+\frac{\delta^2+\delta(1+\beta)}{1-\beta}=\frac{(1+\beta)\frac{\beta\delta}{1+\delta}+[\delta^2+\delta(1+\beta)](1-\beta)}{(1-\beta)^2}=$$

$$=\frac{\beta^2(\frac{\delta}{1+\delta}-\delta)+\beta(\frac{\delta}{1+\delta}-\delta^2)+\delta^2+\delta}{(1-\beta)^2}\le\frac{\beta\frac{\delta}{1+\delta}+\delta^2+\delta}{(1-\beta)^2}<$$

$$<\frac{\frac{\delta}{1+\delta}+\delta^2+\delta}{(1-\beta)^2}\le\frac{\delta^2+2\delta}{(1-\beta)^2}$$

which implies

$$\frac{|||x_n'-y'|||}{1-|||x'|||}\ge\frac{\|x_n-y\|}{1-\|x\|}-\frac{\delta^2+2\delta}{(1-\beta)^2}.$$

Hence

$$\zeta_Y(\beta)\ge\zeta_X(\beta)-\frac{\delta^2+2\delta}{(1-\beta)^2}.$$

A symmetric argument yields $\zeta_X(\beta)\ge\zeta_Y(\beta)-(\delta^2+2\delta)/(1-\beta)^2$.

Next we give the characterizations of nearly uniformly convex and nearly uniformly smooth spaces.

Lemma 3.9. *Let X be a Banach space. Then*

$$\zeta_X(\beta)\ge\frac{(\beta/2)\Delta_0(X)+\beta-1}{1-\beta}$$

Proof. The inequality is obvious if $\Delta_0(X)=0$. If $\Delta_0(X)>0$, assume $0<c<\Delta_0(X)$. For any $\eta>0$ there exists a sequence $\{z_n\}$ in B_X weakly convergent, say to z, such that $\|z_n-z\|\ge c$ and $\|z\|\ge 1-\eta$. Consider the sequence $x_n=\frac{\beta}{2}(z_n+z)\in B_\beta$ which is weakly convergent to $x=\beta z$ with $\liminf_n\|x_n-x\|\le\beta$. Then we have

$$\|x_n-y\|\ge\|x_n-x\|-\|x-y\|\ge\frac{\beta}{2}c+\|x\|-1\ge\frac{\beta}{2}c+\beta(1-\eta)-1.$$

Hence

$$\frac{\|x_n-y\|}{1-\|x\|}\ge\frac{(\beta/2)c+\beta(1-\eta)-1}{1-\beta(1-\eta)}.$$

Letting $\eta\to 0$ and later $c\to\Delta_0(X)$ we obtain the inequality as stated.

Lemma 3.10. *Let X be a Banach space. Then*
$$\frac{1}{2}\Delta_0(X) \le \liminf_{\beta\to 1}(1-\beta)\zeta_X(\beta) \le \limsup_{\beta\to 1}(1-\beta)\zeta_X(\beta) \le 2\Delta_0(X).$$

Proof. From Lemma **??**, we know that $\liminf_{\beta\to 1}(1-\beta)\zeta_X(\beta) \ge 1/2\Delta_0(X)$. On the other hand, consider any sequence $\{x_n\}$ in B_β w-convergent to x with $\liminf_n \|x_n - x\| \le \beta$. Taking a subsequence, if necessary, we can assume that $\lim_n \|x_n - y\|$, $\lim_n \|x_n\| = \alpha \le \beta$ and $\lim_n \|x_n - x\| = \epsilon \le 2\alpha$ do exist. For an arbitrary $\eta \in (0, 1-\beta)$ we can assume $|\|x_n - x\| - \epsilon| < \eta$ and $|\|x_n\| - \alpha| < \eta$ for every n. Choose $p \in (0,1)$ and define for $\beta \in (0,1)$
$$r_p(\beta) = \sup\left\{\epsilon \ge 0 : \Delta_X(\epsilon) < (1-\beta)^p\right\}.$$
It is clear that $r_p(\cdot)$ is nonincreasing and we claim that $\lim_{\beta\to 1^-} r_p(\beta) \le \Delta_0(X)$. Indeed, otherwise there is a number c such that $r_p(\beta) > c > \Delta_0(X)$ for any $\beta < 1$. Choosing $\varepsilon(\beta) \in (c, r_p(\beta))$ such that $\Delta_X(\varepsilon(\beta)) < (1-\beta)^p$, we obtain $\Delta_X(c) \le (1-\beta)^p$ for any $\beta < 1$, which implies $\Delta_X(c) = 0$ and this is a contradiction because $c > \Delta_0(X)$.
We have the following inequalities
$$\|x_n - y\| \le \|x_n - x\| + \|x - y\| \le \epsilon + \eta + 1 - \|x\|$$
$$\|x\| \le (\alpha+\eta)\left(1 - \Delta_X\left(\frac{\epsilon}{\alpha+\eta}\right)\right) \le (\alpha+\eta)(1-\Delta_X(\epsilon))$$
$$\|x\| \ge \|x_n\| - \|x_n - x\| \ge \alpha - 2\eta - \epsilon.$$
Thus
$$\frac{\|x_n - y\|}{1-\|x\|} \le \frac{1-\alpha+3\eta+2\epsilon}{1-\alpha(1-\Delta_X(\epsilon))}.$$
If $\epsilon > r_p(\beta)$, then $\Delta_X(\epsilon) \ge (1-\beta)^p$ and
$$\frac{\|x_n - y\|}{1-\|x\|} \le \frac{1-\alpha+3\eta+4\alpha}{1-\beta(1-\Delta_X(\epsilon))} \le \frac{1+3\alpha+3\eta}{1-\beta+\beta(1-\beta)^p} \le \frac{1+3\beta+3\eta}{1-\beta+\beta(1-\beta)^p}.$$
If $\epsilon \le r_p(\beta)$, we have
$$\frac{\|x_n - y\|}{1-\|x\|} \le \frac{1-\alpha+3\eta+2r_p(\beta)}{1-\alpha} \le \frac{1-\beta+3\eta+2r_p(\beta)}{1-\beta}.$$
In all cases, we have
$$\limsup_{\beta\to 1}(1-\beta)\zeta_X(\beta) \le$$
$$\le \max\left\{\limsup_{\beta\to 1}(1-\beta)\frac{1+3\beta+3\eta}{1-\beta+\beta(1-\beta)^p}, \limsup_{\beta\to 1} 1-\beta+3\eta+2r_p(\beta)\right\} \le$$

$$\leq \max\left\{0, \limsup_{\beta\to 1} 3\eta + 2r_p(\beta)\right\} \leq 2\Delta_0(X) + 3\eta.$$

Since η is arbitrary we obtain $\limsup_{\beta\to 1}(1-\beta)\zeta_X(\beta) \leq 2\Delta_0(X)$

Theorem 3.11. *Let X be a Banach space. Then X is NUC if and only if X is reflexive and* $\lim_{\beta\to 1}(1-\beta)\zeta_X(\beta) = 0$.

Theorem 3.12. *Let X be a Banach space. Then X is NUS if and only if X is reflexive and $\zeta'_X(0) = 0$.*

Proof. Assume that X is NUS and choose $\epsilon > 0$ arbitrary and $\eta = \eta(\epsilon)$ given by the definition of NUS. Take $\beta < \eta/(1+\eta)$. If $\{x_n\}$ is a sequence in B_β which converges weakly to x with $\liminf_n \|x_n - x\| \leq \beta < \eta$. We can assume, without loss of generality, that $\lim_n \|x - x_n\|/(1-\|x\|) =: t$ exists. We have

$$\|y - x_n\| = \|y - x\| \left\| \frac{y-x}{\|y-x\|} + \frac{x - x_n}{\|y-x\|} \right\| =$$

$$= (1-\|x\|)\left\| \frac{y-x}{\|y-x\|} + \frac{\|x-x_n\|}{1-\|x\|}\frac{x-x_n}{\|x-x_n\|} \right\| \leq$$

$$\leq (1-\|x\|)\left\| \frac{y-x}{\|y-x\|} + t\frac{x-x_n}{\|x-x_n\|} \right\| + (1-\|x\|)\left| t - \frac{\|x-x_n\|}{1-\|x\|} \right|.$$

Since

$$t = \lim_n \frac{\|x-x_n\|}{1-\|x\|} \leq \frac{\beta}{1-\beta} < \eta,$$

we have

$$\|y-x_n\| \leq (1-\|x\|)(1+\epsilon t) + (1-\|x\|)\left| t - \frac{\|x-x_n\|}{1-\|x\|} \right| \leq$$

$$\leq (1-\|x\|)\left(1 + \frac{\epsilon\beta}{1-\beta} + \left| t - \frac{\|x-x_n\|}{1-\|x\|} \right|\right)$$

for infinitely many n. Hence

$$\zeta_X(\beta) \leq 1 + \frac{\epsilon\beta}{1-\beta}$$

which implies

$$0 \leq \frac{\zeta_X(\beta) - 1}{\beta} \leq \frac{\epsilon}{1-\beta} \qquad \text{for } \beta < \frac{\eta}{1+\eta}.$$

Thus

$$0 \leq \liminf_{\beta\to 0} \frac{\zeta_X(\beta)-1}{\beta} \leq \limsup_{\beta\to 0} \frac{\zeta_X(\beta)-1}{\beta} \leq \epsilon.$$

Since ϵ is arbitrary we obtain $\zeta'_X(0) = 0$.

Conversely, assume that X is not NUS. There exists $\epsilon_0 > 0$ such that for every $\eta > 0$ there exists $t \in (0, \eta)$ and a weakly null sequence $\{u_n\}$ in B_X satisfying

$\|u_1+tu_n\| > 1+\epsilon_0 t$ for all n. Choose $\eta > 0$ arbitrary and $a = \beta\epsilon_0/2$ with β small enough such that $t = a/(1-a)$. Consider the sequence $x_n = a(u_1 - u_n)/\|u_1\|$ which converges weakly to $au_1/\|u_1\| =: x$. Since

$$1 + \epsilon_0 \frac{a}{1-a} < \left\| u_1 + \frac{a}{1-a} u_n \right\| \leq \|u_1\| + \frac{a}{1-a}$$

and $a/(1-a) < 1$ we obtain

$$\|u_1\| \geq 1 + (\epsilon_0 - 1)\frac{a}{1-a} > \epsilon_0.$$

Thus $\|x_n\| \leq 2a/\epsilon_0 = \beta$, $\liminf_n \|x_n - x\| \leq \beta$ and

$$\|y - x_n\| = \|y - x\| \left\| \frac{y-x}{\|y-x\|} + \frac{x - x_n}{\|y-x\|} \right\| = \|y - x\| \left\| \frac{u_1}{\|u_1\|} + \frac{a}{1-a}\frac{u_n}{\|u_1\|} \right\| \geq$$

$$\geq \frac{\|y-x\|}{\|u_1\|}\left(1 + \epsilon_0 \frac{a}{1-a}\right) \geq (1 - \|x\|)\left(1 + \epsilon_0 \frac{a}{1-a}\right).$$

Hence $\zeta_X(\beta) \geq 1 + \epsilon_0 a/(1-a)$ which implies $(\zeta_X(\beta) - 1)/\beta \geq \epsilon_0^2/(2 - \beta\epsilon_0)$ and

$$\liminf_{\beta\to 0} \frac{\zeta_X(\beta) - 1}{\beta} \geq \frac{\epsilon_0^2}{2} > 0.$$

We can give a lower estimate for $WCS(X)$ depending of the value of $\zeta_X(\beta)$.

Proposition 3.13. *Let X be a Banach space. Then*

$$WCS(X) \geq \sup_{\beta\in(0,1)} \frac{\beta(2 + \zeta_X(\beta))}{\zeta_X(\beta) + 1}$$

Proof. Denote $w = WCS(X)$. For any $\eta > 0$ there exists a weakly null sequence $\{z_n\}$ in B_X such that $\|z_n - z_m\| \leq 1 + \eta$ and $\|z_n\| \geq 1/w(1-\eta)$. Fix $k \in \mathbb{N}$ and for any $\beta \in (0,1)$ we consider the sequence $x_n = \beta(z_k - z_n)/(1+\eta) \in B_\beta$ which converges weakly to $\beta z_k/(1+\eta) =: x$ with $\liminf_n \|x_n - x\| \leq \beta$. Then we have

$$\frac{\|x_n - y\|}{1 - \|x\|} \geq \frac{\frac{\beta}{1+\eta}\|z_n\| - \|z_k\|\left(\frac{1}{\|z_k\|} - \frac{\beta}{1+\eta}\right)}{1 - \frac{\beta}{w(1-\eta^2)}} \geq \frac{\frac{2\beta}{w(1-\eta^2)} - 1}{1 - \frac{\beta}{w(1-\eta^2)}}$$

Letting $\eta \to 0$ we obtain

$$\zeta_X(\beta) \geq \frac{2\beta - w}{w - \beta}$$

which is equivalent to

$$w \geq \frac{\beta(2 + \zeta_X(\beta))}{\zeta_X(\beta) + 1}.$$

Remark: The above lower bound for $WCS(X)$ is not sharp. Indeed, for $X = \ell_1$ we obtain

$$2 = WCS(X) \geq \sup_{\beta \in (0,1)} \frac{\beta(2 + \zeta_X(\beta))}{\zeta_X(\beta) + 1} = \frac{3}{2}$$

and for $X = \ell_2$ elementary calculus proves that

$$\sqrt{2} = WCS(X) \geq \sup_{\beta \in (0,1)} \frac{\beta(2 + \zeta_X(\beta))}{\zeta_X(\beta) + 1} \approx 1.19.$$

However, for $X = c_0$ we have the equality

$$WCS(X) = 1 = \sup_{\beta \in (0,1)} \frac{\beta(2 + \zeta_X(\beta))}{\zeta_X(\beta) + 1}.$$

As a consequence of the previous proposition and having in mind Theorem **??** and Theorem **??** we obtain a sufficient condition so that a Banach space X has w-UNS.

Corollary 3.14. *Let X be a Banach space. If*

$$\zeta_X(\beta) < \frac{2\beta - 1}{1 - \beta} \quad \textit{for some } \beta \in (0,1)$$

in particular if $\liminf_{\beta \to 1}(1-\beta)\zeta_X(\beta) < 1$*, then X has w-UNS and, so, X has the w-FPP.*

Remark: We do not know if the estimate in Corollary **??** is sharp. Notice that, since $\zeta_X(\beta) \geq 1$, this inequality only can be satisfied for $\beta > 2/3$. This fact is not surprising because we have shown that the "good" behaviour of ζ for β near zero is related to nearly uniform smoothness (Theorem **??**) and this property does not imply weak normal structure (see [**?**]).
On the other hand, we cannot hope the estimation $\zeta_X(\beta) < 1/(1-\beta)$ for some β as a warranty for weak uniform normal structure as it happened with the finite-dimensional modulus ξ ([**?**], Proposition 2.9). Indeed, the space $X = \ell_{2,\infty}$ does not have w-UNS and satisfies $\zeta_X(\beta) < 1/(1-\beta)$ for some $\beta \in (0,1)$ because otherwise $\zeta_X(\beta) = 1/(1-\beta)$ for all β which implies $\zeta_X'(0) = 1$ and this is a contradiction because X is NUS.

Finally, we give a sufficient condition for Banach space X to have the w-FPP in absence of w-NS. We shall use the coefficient $R(X)$.

Theorem 3.15. *Let X be a Banach space. If $\zeta_X(\beta) < 1+\beta$ for some $\beta \in (0,1)$, in particular if $\zeta_X'(0) < 1$, then $R(X) < 2$ and, so, X has the w-FPP.*

Proof. Assume $R(X) = 2$ and choose $\eta > 0$ arbitrary. There exists a weakly null sequence $\{u_n\}$ in B_X and a point $u \in B_X$ such that $\liminf_n \|u_n + u\| > 2 - \eta$. We can assume $\|u_n\| > 1 - \eta$ for every n and $\|u\| > 1 - \eta$. It is clear that

$\|\lambda u_n + \mu u\| \geq (\lambda+\mu)(1-\eta)$ for every $\lambda, \mu \in (0, +\infty)$ and every n. Consider the sequence

$$x_n = \beta\left(\frac{1}{1+a}u - \frac{a}{1+a}u_n\right) \rightharpoonup \frac{\beta}{1+a}u = x$$

with $a > 0$ arbitrary. We have

$$\|y - x_n\| = \left\|\left(\frac{1}{\|u\|} - \frac{\beta}{1+a}\right)u + \frac{\beta a}{1+a}u_n\right\| \geq$$

$$\geq \left(\frac{1}{\|u\|} - \frac{\beta}{1+a} + \frac{\beta a}{1+a}\right)(1-\eta) \geq \left(1 - \frac{\beta}{1+a} + \frac{\beta a}{1+a}\right)(1-\eta)$$

$$\|x\| = \frac{\beta}{1+a}\|u\| > \frac{\beta}{1+a}(1-\eta).$$

Hence

$$\frac{\|x_n - y\|}{1-\|x\|} \geq \frac{\left(1 - \frac{\beta}{1+a} + \frac{\beta a}{1+a}\right)(1-\eta)}{1 - \frac{\beta}{1+a}(1-\eta)}.$$

Letting $\eta \to 0$ and $a \to \infty$, we obtain $\zeta_X(\beta) \geq 1+\beta$ for all $\beta \in (0,1)$.

Remarks:

(1) If X is a reflexive Banach space with $\zeta'_X(0) < 1$ then, as a consequence of the previous theorem and [**?**] Corollary 4.3.6., we have $\Gamma'_X(0) < 1$, where $\Gamma_X(\cdot)$ denotes the modulus of NUS of X (see [**?**]).

(2) We do not know if $R(X)$ can be less than 2 for some space X satisfying $\zeta_X(\beta) \geq 1+\beta$ for all β.

(3) The estimate in Theorem **??** does not imply new fixed point results for β close to 1. Indeed, if $\beta \geq -1+\sqrt{3}$ we have

$$\frac{2\beta - 1}{1-\beta} \geq 1+\beta.$$

In this sense, we can say that Corollary **??** is useful for β close to 1 and Theorem **??** for β close to 0. It would be interesting to obtain fixed point results from the modulus $\zeta_X(\beta)$ for medium values of β in some spaces where $WCS(X) = 1$ and $R(X) = 2$.

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