Let $X$ be a Banach space whose characteristic of noncompact convexity is less than 1 and satisfies the non-strict Opial condition. Let $C$ be a bounded closed convex subset of $X$, $KC(C)$ the family of all compact convex subsets of $C$ and $T$ a nonexpansive mapping from $C$ into $KC(C)$. We prove that $T$ has a fixed point. The non-strict Opial condition can be removed if, in addition, $T$ is an $1-\chi$-contractive mapping.

1. Introduction

Some classical fixed point theorems for singlevalued nonexpansive mappings have been extended to multivalued mappings. The first results in this direction were established by J. Markin [12] in a Hilbert space setting and by F. Browder [3] for spaces having a weakly continuous duality mapping. E. Lami Dozo [9] generalized these results to a Banach space satisfying Opial’s condition.

By using Edelstein’s method of asymptotic centers, T.C. Lim [10] obtained a fixed point theorem for a multivalued nonexpansive self-mapping in a uniformly convex Banach space. W. A. Kirk and S. Massa [7] gave an extension of Lim’s theorem proving the existence of a fixed point in a Banach space for which the asymptotic center of a bounded sequence in a closed bounded convex subset is nonempty and compact.

Many questions remain open (see [15] and [16]) about the existence of fixed points for multivalued nonexpansive mappings when the Banach space satisfies geometric properties which assure the existence of a fixed point in the singlevalued case, for instance, if $X$ is a nearly uniformly convex space. In this paper we state some fixed point theorems for multivalued nonexpansive self-mappings, which are more general than the previous results. First, we give a fixed point theorem for a multivalued nonexpansive and $1-\chi$-contractive mapping in the framework of a Banach space.
whose characteristic of noncompact convexity associated to the separation measure of noncompactness is less than 1. If, in addition, the space satisfies the non-strict Opial condition, we prove, using some properties of $\chi$-minimal sets (see [2, Chapter III] for definitions), that the $\chi$-contractiveness assumption can be removed. In particular, this result gives a partial answer to the above open question.

2. PRELIMINARIES AND NOTATIONS

Let $X$ be a Banach space. We denote by $CB(X)$ the family of all nonempty closed bounded subsets of $X$ and by $K(X)$ (resp. $KC(X)$) the family of all nonempty compact (resp. compact convex) subsets of $X$. On $CB(X)$ we have the Hausdorff metric $H$ given by

$$H(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, \quad A, B \in CB(X)$$

where for $x \in X$ and $E \subset X$ $d(x, E) := \inf \{d(x, y) : y \in E\}$ is the distance from the point $x$ to the subset $E$.

If $C$ is a closed convex subset of $X$, then a multivalued mapping $T : C \rightarrow CB(X)$ is said to be a contraction if there exists a constant $k \in [0, 1)$ such that $H(Tx, Ty) \leq k\|x - y\|$, $x, y \in C$, and $T$ is said to be nonexpansive if $H(Tx, Ty) \leq \|x - y\|$, $x, y \in C$.

Recall that the Kuratowski and Hausdorff measures of noncompactness of a nonempty bounded subset $B$ of $X$ are respectively defined as the numbers:

$$\alpha(B) = \inf \{d > 0 : B \text{ can be covered by finitely many sets of diameter } \leq d\},$$

$$\chi(B) = \inf \{d > 0 : B \text{ can be covered by finitely many balls of radius } \leq d\}.$$

Then a multivalued mapping $T : C \rightarrow 2^X$ is called $\gamma$-condensing (resp. $1-$\-$\gamma$-contractive) where $\gamma = \alpha(\cdot)$ or $\chi(\cdot)$ if, for each bounded subset $B$ of $C$ with $\gamma(B) > 0$, there holds the inequality

$$\gamma(T(B)) < \gamma(B) \quad (\text{resp. } \gamma(T(B)) \leq \gamma(B)).$$

Here $T(B) = \cup_{x \in B} Tx$. Note that a multivalued mapping $T : C \rightarrow 2^X$ is said to be upper semicontinuous on $C$ if $\{x \in C : Tx \subset V\}$ is open in $C$ whenever $V \subset X$ is open; $T$ is said to be lower semicontinuous if $T^{-1}(V) := \{x \in C : Tx \cap V \neq \emptyset\}$ is open in $C$ whenever $V \subset X$ is open; and $T$ is said to be continuous if it is both upper and lower semicontinuous. There is another different kind of continuity for set-valued operators: $T : X \rightarrow CB(X)$ is said to be continuous on $X$ (with respect to the Hausdorff metric $H$) if $H(Tx_n, Tx) \rightarrow 0$ whenever $x_n \rightarrow x$. It is not hard to see (see [1] and [5]) that both definitions of continuity are equivalent if $Tx$ is compact for every $x \in X$. We say that $x \in C$ is a fixed point of $T$ if and only if $x$ is contained in $Tx$.

In the next section we shall use the following result for multivalued mappings (see also [14]).
Theorem 2.1 ([4]). Let $X$ be a Banach space and $\emptyset \neq D \subset X$ be closed bounded convex. Let $F : D \to 2^X$ be upper semicontinuous $\gamma$-condensing with closed convex values, where $\gamma(\cdot) = \alpha(\cdot)$ or $\chi(\cdot)$. If $Fx \cap I_D(x) \neq \emptyset$ on $D$ then $\text{Fix}(F) \neq \emptyset$. (Here $I_D(x)$ is called the inward set at $x$ defined by $I_D(x) := \{ x + \lambda (y - x) : \lambda \geq 0, y \in D \}$).

Let us recall some definitions of properties satisfied by a Banach space $X$:

Definition 2.1. (a) $X$ is said to be nearly uniformly convex (NUC) if it is reflexive and its norm is uniformly Kadec-Klee, that is, for any positive number $\epsilon$ there exists a corresponding number $\delta = \delta(\epsilon) > 0$ such that for any sequence $\{x_n\}$

$$\|x_n\| \leq 1 \quad n = 1, 2, \ldots$$

$$w - \lim_n x_n = x$$

$$\text{sep}(\{x_n\}) = \inf \{\|x_n - x_m\| : n \neq m\} \geq \epsilon$$

$$\implies \|x\| \leq 1 - \delta.$$ 

(b) $X$ is said to satisfy the Opial condition if, whenever a sequence $\{x_n\}$ in $X$ converges weakly to $x$, then for $y \neq x$

$$\limsup_n \|x_n - x\| < \limsup_n \|x_n - y\|.$$ 

If the inequality is non strict we say that $X$ satisfies the non-strict Opial condition.

3. ASYMPTOTIC CENTERS AND MODULI OF NONCOMPACT CONVEXITY

In this section we shall consider, apart from $\alpha$ and $\chi$, another measure of noncompactness. The separation measure of noncompactness of a nonempty bounded subset $B$ of $X$ is defined by

$$\beta(B) = \sup \{\epsilon : \text{ there exists a sequence } \{x_n\} \in B \text{ such that } \text{sep}(\{x_n\}) \geq \epsilon\}.$$ 

Definition 3.1. Let $X$ be a Banach space and $\phi = \alpha$, $\beta$ or $\chi$. The modulus of noncompact convexity associated to $\phi$ is defined in the following way

$$\Delta_{X, \phi}(\epsilon) = \inf \{1 - d(0, A) : A \subset B_X \text{ is convex, } \phi(A) \geq \epsilon\}.$$ 

($B_X$ is the unit ball of $X$).

The characteristic of noncompact convexity of $X$ associated with the measure of noncompactness $\phi$ is defined by

$$\epsilon_{\phi}(X) = \sup \{\epsilon \geq 0 : \Delta_{X, \phi}(\epsilon) = 0\}.$$ 

The following relationships among the different moduli are easy to obtain

$$\Delta_{X, \alpha}(\epsilon) \leq \Delta_{X, \beta}(\epsilon) \leq \Delta_{X, \chi}(\epsilon),$$

and consequently

$$\epsilon_{\alpha}(X) \geq \epsilon_{\beta}(X) \geq \epsilon_{\chi}(X).$$
When $X$ is a reflexive Banach space we have some alternative expressions for
the moduli of noncompact convexity associated with $\beta$ and $\chi$,

$$
\Delta_{X,\beta}(\epsilon) = \inf \{ 1 - \|x\| : \{x_n\} \subset B_X, \ x = w - \lim_n x_n, \ \text{sep}(\{x_n\}) \geq \epsilon \},
$$

$$
\Delta_{X,\chi}(\epsilon) = \inf \{ 1 - \|x\| : \{x_n\} \subset B_X, \ x = w - \lim_n x_n, \ \chi(\{x_n\}) \geq \epsilon \}.
$$

It is known that $X$ is NUC if and only if $\epsilon_\phi(X) = 0$, where $\phi$ is $\alpha$, $\beta$ or $\chi$. The above-mentioned definitions and properties can be found in [2].

Let $C$ be a nonempty bounded closed subset of $X$ and $\{x_n\}$ a bounded sequence in $X$, we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in $C$, respectively, i.e.

$$
r(C, \{x_n\}) = \inf \{ \limsup_n \|x_n - x\| : x \in C \},
$$

$$
A(C, \{x_n\}) = \{ x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\}) \}.
$$

It is known that $A(C, \{x_n\})$ is a nonempty weakly compact convex set as $C$ is.

Next, we present a theorem which gives a connection between the asymptotic center of a sequence and the modulus of noncompact convexity and it will play a crucial role in our results. Previously, recall the following notation of regularity and the lemma below.

**Definition 3.2.** Let $\{x_n\}$ and $C$ be as above. Then $\{x_n\}$ is called regular with respect to (w.r.t.) $C$ if $r(C, \{x_n\}) = r(C, \{x_n\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

**Lemma 3.1 (Goebel[6], Lim[11]).** Let $\{x_n\}$ and $C$ be as above. Then, there always exists a subsequence of $\{x_n\}$ which is regular w.r.t. $C$.

If $D$ is a bounded subset of $X$, the Chebyshev radius of $D$ relative to $C$ is defined by

$$
r_C(D) := \inf \{ \sup \|x - y\| : y \in D \} : x \in C \}.
$$

**Theorem 3.1.** Let $C$ be a closed convex subset of a reflexive Banach space $X$ and let $\{x_n\}$ be a bounded sequence in $C$ which is regular w.r.t. $C$. Then

$$
r_C(A(C, \{x_n\})) \leq (1 - \Delta_{X,\beta}(1^-))r(C, \{x_n\}).
$$

Moreover, if $X$ satisfies the non-strict Opial condition then

$$
r_C(A(C, \{x_n\})) \leq (1 - \Delta_{X,\chi}(1^-))r(C, \{x_n\}).
$$

**Proof.** Denote $r = r(C, \{x_n\})$ and $A = A(C, \{x_n\})$. Since $\rho(\{x_n\}) \subset C$ is a weakly compact set, we can find a subsequence $\{y_n\}$ of $\{x_n\}$ weakly convergent to
a point \( z \in C \). Without loss of generality we assume that the limit \( \lim_{n \neq m} \| y_n - y_m \| \) exists (see [2, Theorem III.1.5]). Since \( \{ x_n \} \) is regular w.r.t. \( C \), \( r = r(C, \{ y_n \}) \) and then, the weakly lower semicontinuity of the norm implies

\[
r \leq \limsup_n \| y_n - z \| \leq \liminf_m \limsup_n \| y_n - y_m \| = \lim_{n \neq m} \| y_n - y_m \|.
\]

Hence \( \beta(\{ y_n \}) \geq r \).

On the other hand, if \( X \) satisfies the non-strict Opial condition, it is easy to deduce that \( \chi(\{ y_n : n \in \mathbb{N} \}) = \lim sup \| y_n - z \| \). Indeed, for every \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that \( \| y_n - y \| < \lim sup \| y_n - y \| + \epsilon \) for all \( n \geq n_0 \), and hence \( \chi(\{ y_n : n \in \mathbb{N} \}) \leq \lim sup \| y_n - y \| \).

Conversely let us suppose that \( \{ y_n : n \in \mathbb{N} \} \) can be covered by finitely many balls with radius \( r < \lim sup \| y_n - y \| \). Consider a subsequence \( \{ z_{n_k} \} \) of \( \{ y_n \} \) such that \( \lim_{n} \| z_{n_k} - z \| = \lim sup \| y_n - z \| \). Then there exists a subsequence \( \{ z_{n_k} \} \) of \( \{ z_n \} \) contained in a ball \( B(x, r) \) for some \( x \in X \). Therefore we obtain

\[
\limsup_k \| z_{n_k} - x \| \leq r < \lim sup \| y_n - z \| = \lim_k \| z_{n_k} - z \|,
\]

contradicting the fact that \( X \) satisfies the non-strict Opial condition, because \( z_{n_k} \rightharpoonup z \).

Thus, in this case we have \( \chi(\{ y_n : n \in \mathbb{N} \}) \geq r \).

Assume \( x \) lies in \( A \). Since \( r = \lim sup \| y_n - x \| \), for every \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that \( \| y_n - x \| < r + \epsilon \) for all positive integer \( n \) greater than or equal to \( n_0 \). Hence, the sequence

\[
\left\{ \frac{y_n - x}{r + \epsilon} \right\}_{n \geq n_0}
\]

is contained in the unit ball of \( X \), converges weakly to \( \frac{z - x}{r + \epsilon} \) and \( \beta\left( \left\{ \frac{y_n - z}{r + \epsilon} \right\} \right) \geq \frac{r}{r + \epsilon} \).

If \( X \) satisfies the non-strict Opial condition \( X \) we also have that \( \chi\left( \left\{ \frac{y_n - z}{r + \epsilon} \right\} \right) \geq \frac{r}{r + \epsilon} \). Therefore we deduce

\[
\| x - z \| = \left( 1 - \Delta_{X \times \mathbb{R}} \left( \frac{r}{r + \epsilon} \right) \right) (r + \epsilon),
\]

and in the second assumption

\[
\| x - z \| = \left( 1 - \Delta_{X \times \mathbb{R}} \left( \frac{r}{r + \epsilon} \right) \right) (r + \epsilon).
\]

Since the last inequality is true for every \( \epsilon > 0 \) and for every \( x \in A \), we obtain the inequalities in the statement.
Remark 3.1.- It must be noted that the regularity assumption is necessary in the above theorem. Indeed, consider the product space $X = \ell^2_\infty \otimes \ell_2$, where $\ell^2_\infty := (\mathbb{R}^2, \| \cdot \|_\infty)$, with the norm

$$
\|(x, y)\| = \left(\|x\|_\infty^2 + \|y\|_2^2\right)^{\frac{1}{2}}, \quad x \in \ell^2_\infty, \; y \in \ell_2.
$$

First, we are going to prove that

$$
\Delta_{X,\alpha}(\epsilon) = 1 - \sqrt{1 - \epsilon^2/4}.
$$

Since $X$ contains isometrically $\ell_2$, it is easy to deduce that

$$
\Delta_{X,\alpha}(\epsilon) \leq \Delta_{\ell_2,\alpha}(\epsilon) = 1 - \sqrt{1 - \epsilon^2/4}
$$

(see [2, Chapter I]).

Let us now study the reverse inequality. Taking in mind that $\alpha(A) \leq 2\chi(A)$ ([2]) for each bounded subset of $X$, it is clear that

$$
\Delta_{X,\alpha}(\epsilon) \geq \Delta_{X,\chi}\left(\frac{\epsilon}{2}\right),
$$

for all $\epsilon > 0$. Let us estimate the value of $\Delta_{X,\chi}\left(\frac{\epsilon}{2}\right)$. Since $X$ is reflexive, we have (see [2, Chapter V])

$$
\Delta_{X,\chi}\left(\frac{\epsilon}{2}\right) = \inf\{1 - \|z\| : w - \lim_n z_n = z, \; \|z_n\| \leq 1, \; \chi(\{z_n\}) \geq \frac{\epsilon}{2}\}.
$$

Let $\{(x_n, y_n)\}$ be a sequence in the unit ball of $X$ weakly convergent to a vector $(x_o, y_o) \in X$ such that $\chi(\{(x_n, y_n)\}) \geq \frac{\epsilon}{2}$.

It follows that $\lim_n x_n = x_o$ and $\{y_n\}$ is weakly convergent to $y_o$ in $\ell_2$. Taking a subsequence if necessary, we can assume that $\lim_n \|y_n - y_o\|_2$ and $\lim_n \|y_n\|_2$ exist, and the supports of $y_n - y_o$ and $y_o$ are nearly disjoint, that is

$$
\lim_n \|y_n\|_2^2 = \|y_o\|_2^2 + \lim_n \|y_n - y_o\|_2^2.
$$

On the other hand, it is not difficult to see that $X$ satisfies the Opial condition. In fact, it satisfies the uniform Opial condition with the same modulus of Opial associated with $\ell_2$. Then

$$
\chi(\{(x_n, y_n)\}) = \limsup_n \|(x_n, y_n) - (x_o, y_o)\| = \lim_n \|y_n - y_o\|_2 \geq \frac{\epsilon}{2}.
$$

Hence

$$
1 \geq \lim_n \|(x_n, y_n)\|^2 = \lim_n \|x_n\|_\infty^2 + \|y_n\|_2^2
$$

$$
= \|x_o\|_\infty^2 + \|y_o\|_2^2 + \lim_n \|y_n - y_o\|_2^2
$$

$$
= \|(x_o, y_o)\|^2 + \lim_n \|y_n - y_o\|_2^2
$$

$$
\geq \|(x_o, y_o)\|^2 + \frac{\epsilon^2}{4}.
$$
Thus
\[ \Delta_{X,\chi}(\varepsilon_2) \geq 1 - \sqrt{1 - \frac{\varepsilon_2^2}{4}}, \]
following the required inequality.

Moreover, since \( X \) is reflexive and satisfies the uniform Opial condition then \( \Delta_{X,\chi}(1^-) = 1 \) (see Chapter V in [2] for details).

If \( x_n \in \mathbb{R}^2 \) is the sequence defined by \( x_{2n-1} = (-1, 0) \) and \( x_{2n} = (1, 0) \) for each \( n \in \mathbb{N} \), we consider the sequence \( z_n = (x_n, 0) \in X \).

Denote \( B \) the unit ball of \( \ell_\infty^2 \) and let \( C = B \times \{0\} \). Clearly \( C \) is a weakly compact convex subset of \( X \) which contains \( \{z_n\} \). It is not difficult to see that \( r(C, \{z_n\}) = 1 \) and \( A(C, \{z_n\}) = \{(0, y), 0 : y \in [-1,1]\} \). Then \( r_C(A(C, \{z_n\})) = 1 \), while \( 1 - \Delta_{X,\alpha}(1^-) = \sqrt{\frac{3}{2}} \) and \( 1 - \Delta_{X,\chi}(1^-) = 0 \) are less than one.

4. Fixed point theorems

In order to prove our first result, we need the following proposition which is proved along the proof of the Kirk-Massa theorem as it appears in [16].

**Proposition 4.1.** Let \( C \) be a nonempty weakly compact and separable subset of a Banach space \( X \), \( T : C \to K(C) \) a nonexpansive mapping and \( \{x_n\} \) a sequence in \( C \) such that \( \lim d(x_n, Tx_n) = 0 \). Then, there exists a subsequence \( \{z_n\} \) of \( \{x_n\} \) such that \( Tx \cap A \neq \emptyset, \ \forall x \in A := A(C, \{z_n\}) \).

Assume that \( C \) is a nonempty weakly compact convex subset of a Banach space \( X \) and \( T : C \to KC(C) \) is a nonexpansive and 1-\( \chi \)-contractive self-mapping. Consider a bounded sequence \( \{x_n\} \) in \( C \) such that \( T \) satisfies the condition
\[ Tx \cap A \neq \emptyset, \ \forall x \in A := A(C, \{x_n\}). \]

For a fixed element \( x_0 \in A \) and an arbitrary \( \mu \in (0, 1] \), the contraction \( T_\mu : A \to KC(C) \) defined by
\[ T_\mu x = \mu x_0 + (1 - \mu)Tx, \quad x \in A \]
verifies the hypotheses of Theorem 2.1. Indeed, let \( B \) be a bounded and nonprecompact subset of \( C \). Since \( T \) is 1-\( \chi \)-contractive and \( T_\mu(B) = \mu x_0 + (1 - \mu)T(B) \) we have
\[
\chi(T_\mu(B)) = \chi(\mu x_0 + (1 - \mu)T(B)) = \chi((1 - \mu)T(B)) = (1 - \mu)\chi(T(B)) \leq (1 - \mu)\chi(B) < \chi(B).
\]
Thus \( T_\mu \) is \( \chi \)-condensing. Moreover, since \( A \) is convex, \( T_\mu \) satisfies the same boundary condition as \( T \) does, i.e. we have
\[ T_\mu x \cap A \neq \emptyset, \ \forall x \in A. \]
Hence by Theorem 2.1 $T_{\mu}$ has a fixed point $z_{\mu} \in A$ and we can find a sequence $\{z_n\}$ in $A$ satisfying $\lim n d(z_n, Tz_n) = 0$. Notice that this conclusion is true for every $A$ closed bounded convex subset of $C$ satisfying $Tx \cap A \neq \emptyset$, $\forall x \in A$.

With this observation we are able to prove our main result.

**Theorem 4.1.** Let $C$ be a nonempty closed bounded convex subset of a Banach space $X$ such that $\epsilon_{\beta}(X) < 1$, and $T : C \to KC(C)$ be a nonexpansive and $1-$contractive nonexpansive mapping. Then $T$ has a fixed point.

**Proof.** Let $x_0 \in C$ be fixed and, for each $n \geq 1$, define

$$T_n x := \frac{1}{n} x_0 + (1 - \frac{1}{n}) Tx, \quad x \in C.$$  

Then $T_n$ is a multivalued contraction and hence has a fixed point $x_n$ by Nadler’s theorem ([13]). It is easily seen that $\lim n d(x_n, Tx_n) = 0$. By Lemma 3.1 we may assume that $\{x_n\}$ is regular w.r.t. $C$ and using Proposition 4.1 we can also assume that $Tx \cap A \neq \emptyset$, $\forall x \in A := A(C, \{x_n\})$.

Since condition $\epsilon_{\beta}(X) < 1$ implies reflexivity [2], we apply Theorem 3.1 to obtain

$$r_C(A) \leq \lambda r(C, \{x_n\}),$$

where $\lambda := 1 - \Delta_{X, \beta}(1^{-}) < 1$.

According to the previous observation, we can take a sequence $\{x_n^1\}$ in $A$ satisfying $\lim n d(x_n^1, Tx_n^1) = 0$ and again reasoning as above we can assume that $\{x_n^1\}$ is regular w.r.t. $C$ and

$$Tx \cap A^1 \neq \emptyset, \quad \forall x \in A^1 := A(C, \{x_n^1\}).$$

Again applying Theorem 3.1 we obtain

$$r_C(A^1) \leq \lambda r(C, \{x_n^1\}).$$

On the other hand, since $\{x_n^1\} \subset A$

$$r(C, \{x_n^1\}) \leq r_C(A)$$

and then

$$r_C(A^1) \leq \lambda r_C(A).$$

By induction, for each $m \geq 1$ we construct $A^m$ and $\{x_n^m\}_n$ where $A^m = A(C, \{x_n^m\})$, $\{x_n^m\}_n \subset A^{m-1}$ such that $\lim n d(x_n^m, Tx_n^m) = 0$ and

$$r_C(A^m) \leq \lambda^m r_C(A).$$

Choose $x_m \in A^m$. We shall prove that $\{x_m\}_m$ is a Cauchy sequence. For each $m \geq 1$ we have for any positive integer $n$

$$\|x_{m-1} - x_m\| \leq \|x_{m-1} - x_n^m\| + \|x_n^m - x_m\| \leq \text{diam} A^{m-1} + \|x_n^m - x_m\|.$$
MULTIVALUED NONEXPANSIVE MAPPINGS

Taking upper limit as \( n \to \infty \)

\[
\|x_{m-1} - x_m\| \leq \text{diam}A^{m-1} + \limsup_n \|x_n^m - x_m\| = \text{diam}A^{m-1} + r(C, \{x_n^m\}) \\
\leq \text{diam}A^{m-1} + r_C(A^{m-1}) \\
\leq 2r_C(A^{m-1}) + 3r_C(A) \leq 3\lambda^{m-1}r_C(A).
\]

Since \( \lambda < 1 \), we conclude that there exists \( x \in C \) such that \( x_m \) converges to \( x \).

Let us see that \( x \) is a fixed point of \( T \). For each \( m \geq 1 \),

\[
d(x_m, Tx_m) \leq \|x_m - x_n^m\| + d(x_n^m, Tx_n^m) + H(Tx_n^m, Tx_m) \leq 2\|x_m - x_n^m\| + d(x_n^m, Tx_n^m).
\]

Taking upper limit as \( n \to \infty \)

\[
d(x_m, Tx_m) \leq 2\limsup_n \|x_m - x_n^m\| \leq 2\lambda^{m-1}r_C(A).
\]

Finally, taking limit in \( m \) in both sides we obtain \( \lim_m d(x_m, Tx_m) = 0 \) and the continuity of \( T \) implies that \( d(x, Tx) = 0 \) i.e. \( x \in Tx \).

**Remark 4.1.**- The inductive construction of the sequence \( \{A^m\}_m \) in Theorem 4.1, also appears in [17, Theorem 3.2], though only two steps are done.

**Remark 4.2.**- Note that Theorem 4.1 does not hold if nonexpansiveness assumption is removed. Indeed, if \( B_2 \) is the closed unit ball of \( l_2 \) and \( T : B_2 \to B_2 \) is defined by

\[
T(x) = T(x_1, x_2, ...) = (\sqrt{1 - \|x\|^2}, x_1, x_2, ...),
\]

then \( T \) is an \( 1-\chi \)-contraction without a fixed point.

We do not know if \( \chi \)-contractiveness condition can be dropped in the above theorem. In fact, it is an open problem if every nonexpansive mapping \( T : C \to KC(C) \) is \( 1-\chi \)-contractive even for single-valued mappings. However, if \( X \) is either a separable or a reflexive Banach space and satisfies the non-strict Opial condition this assertion is true, as we prove in the next theorem.

**Theorem 4.2.** If \( X \) is either a separable or reflexive Banach space satisfying the non-strict Opial condition, \( C \) is a nonempty weakly compact subset of \( X \) and \( T : C \to KC(C) \) is a nonexpansive mapping, then \( T \) is \( 1-\chi \)-contractive.

**Proof.** Let \( B \) be an infinite subset of \( C \). Since \( T(B) \) is an infinite and bounded set there exists a sequence \( \{y_n\} \subset T(B) \) which is \( \chi \)-minimal (see [2, Chapter III] for definitions and properties concerning \( \chi \)-minimality). Since \( \chi \) is strictly minimalizable we can assume that

\[
\chi(\{y_n : n \in \mathbb{N}\}) = \chi(T(B)).
\]

Since \( C \) is weakly compact, there is a subsequence of \( \{y_n\} \) which is weakly convergent to some \( y \in C \). Taking a subsequence, if necessary, we can suppose
that \( y_n \to y \) and \( \lim \| y_n - y \| \) exists. As in the proof of Theorem 3.1 we have \( \chi(\{y_n : n \in \mathbb{N}\}) = \lim \| y_n - y \| \).

Choose \( x_n \in B \) such that \( y_n \in T x_n \). Taking a subsequence, if necessary, and following the above argument we assume that \( x_n \to u \in C \), \( \lim \| x_n - u \| \) exists and \( \chi(\{x_n : n \in \mathbb{N}\}) = \lim \| x_n - u \| \).

On the other hand, because \( T \) is compact valued, we can take \( u_n \in Tu \) verifying
\[
\| y_n - u_n \| = d(y_n, Tu) \leq H(T x_n, Tu) \leq \| x_n - u \|, \quad n \geq 1.
\]

By the compactness of \( Tu \), we may assume that \( \{u_n\} \) converges (strongly) to a point \( v \in Tu \). It follows that
\[
\chi(T(B)) = \lim_n \| y_n - y \| \leq \limsup_n \| y_n - v \| = \limsup_n \| y_n - u_n \| \leq \lim_n \| x_n - u \| = \chi(\{x_n\}) \leq \chi(B),
\]
and \( T \) is 1-\( \chi \)-contractive.

In view of this result, we deduce from Theorem 4.1 the following:

**Corollary 4.1.** Let \( X \) be a Banach space with \( \epsilon_\beta(X) < 1 \) which satisfies the non-strict Opial condition. Suppose \( C \) is a nonempty weakly compact convex subset of \( X \) and \( T : C \to KC(C) \) is a nonexpansive mapping. Then \( T \) has a fixed point.

Furthermore, from Theorem 3.1 the method used in the proof of the Theorem 4.1 may be followed to obtain

**Theorem 4.3.** Let \( X \) be a Banach space with \( \epsilon_\chi(X) < 1 \) which satisfies the non-strict Opial condition. Suppose \( C \) is a nonempty weakly compact convex subset of \( X \) and \( T : C \to KC(C) \) is a nonexpansive mapping. Then \( T \) has a fixed point.

This theorem extends the Kirk-Massa theorem, in the sense that we do not need the compactness of asymptotic center of a bounded sequence with respect to a bounded closed convex subset of \( X \). Next example, due to Kuczumow and Prus, illustrates this fact.

**Example.**-([8]) Let \( X_m \) be the space \( \ell_2 \) renormed as follows. For \( x = \sum_{k=1}^{\infty} x_k e_k \) (\( \{e_k\} \) denotes the standard basis in \( \ell_2 \)) set
\[
\| x \|_m = \sup_n \left( x_n^2 + \frac{1}{m+1} \sum_{k=n+1}^{\infty} x_k^2 \right)^{\frac{1}{2}}, \quad m \geq 1.
\]

Clearly \( \| \cdot \|_m \) is equivalent to the usual norm in \( \ell_2 \). \( X_m \) is NUC for each \( m \geq 1 \), and it is easy to check that it satisfies the non-strict Opial condition. Thus, the conclusion of Theorem 4.3 holds for these spaces. However, by non-strict Opial condition we have for any \( x \in X_m \)
\[
\limsup \| x - e_n \| \geq 1,
\]
while for all $k, n$ with $k < n$

$$\| \frac{1}{\sqrt{m+1}} e_k - e_n \|_m = 1.$$ 

Thus we conclude

$$A(X_m, \{e_n\}) \supsetneq \frac{1}{\sqrt{m+1}} \sigma \{e_n\}$$

and, in particular $A(X_m, \{e_n\})$ is not compact.

Note that we cannot apply Lami-Dozo’s theorem [9] to obtain a fixed point because $X_m$ does not satisfy strict Opial condition.

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