CHARACTERIZATION OF KADEC-KLEE PROPERTIES IN ORLICZ SPACE

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Abstract

We study the connection between the Kadec-Klee property for local convergence in measure (H_l), the Kadec-Klee property for global convergence in measure (H_g) and the Δ_2-condition for an Orlicz function space $L^\Phi(\mu)$ equipped with either the Luxemburg norm $\|\cdot\|_\Phi$ or the Orlicz norm $\|\cdot\|^0_\Phi$. Nominally, we prove that the following conditions are equivalent for $(L^\Phi(\mu), \|\cdot\|_\Phi)$:(1) Φ satisfies the suitable Δ_2-condition. (2) $L^\Phi(\mu) \in H_l$.(3) $L^\Phi(\mu) \in H_g$. For $(L^\Phi(\mu), \|\cdot\|^0_\Phi)$ we prove that Φ satisfies the Δ_2-condition at ∞ if $L^\Phi(\mu) \in H_g$. However, an example of an Orlicz space with the Orlicz norm is shown where $L^\Phi(\mu) \in H_g$ but Φ does not satisfy the suitable Δ_2-condition.

1 Introduction

If $(E, \|.\|_E)$ is a normed linear space, then E is said to have the Kadec-Klee property $(E \in H)$ if and only if the sequential weak convergence on the unit sphere coincides with the norm convergence. It is well known that the classical L_p-spaces $1 < p < \infty$ have the Kadec-Klee property (see[**?**],[**?**]). Although the space $L_1[0,1]$ fails to have the Kadec-Klee property, Riesz showed that each sequence almost everywhere convergent on the unit sphere of an L_p-spaces $1 \leq p < \infty$, is also norm-convergent.

Let E be a Banach function space over a measure space (Ω, Σ, μ). E is said to have the Kadec-Klee property for global convergence in measure $(E \in H_g)$, if for all $\{x_n\}$ and x in the unit ball whenever $x_n \to x$ globally in measure on Ω then $\|x_n - x\| \to 0$. E is said to have the Kadec-Klee property for local convergence in measure $(E \in H_l)$, if for all $\{x_n\}$ and x in the unit ball whenever $x_n \to x$ locally in measure on Ω then $\|x_n - x\| \to 0$.

These properties were investigated in [**?**] and [**?**] for symmetric spaces defined on any interval $[0, \alpha), 0 < \alpha \leq \infty$ and on the interval $[0, 1)$, respectively.

In this paper we study the connection between the Kadec-Klee property for local convergence in measure, the Kadec-Klee property for global convergence in measure and the Δ_2-condition for an Orlicz function space equipped with either the Luxemburg norm or the Orlicz norm

We start fixing some notation. In the following R, R^+ and N will stand for the sets of real numbers, nonnegative numbers and positive integers, respectively. By $\Phi : R \to [0, +\infty]$ we shall denote an Orlicz function, i.e., Φ is convex, even, left continuous on the whole R^+, $\Phi(0) = 0$ and Φ is not identically equal to zero. For any Orlicz function Φ we denote

$$a_\Phi := \sup\{u \geq 0 : \Phi(u) = 0\}$$

$$c_\Phi := \sup\{u > 0 : \Phi(u) < +\infty\}.$$

We shall say that an Orlicz function satisfies the Δ_2-condition for all $u \in R$ (at infinity) [at zero] if there are positive constans K ([and u_0 with $0 < \Phi(u_0) < \infty$]) such that $\Phi(2u) \leq K\Phi(u)$ holds for all $u \in R$ (for every $|u| \geq u_0$)[for every $|u| \leq u_0$]. Obviously Φ satisfies the Δ_2-condition for all $u \in R$ if and only if satisfies the Δ_2-condition at zero and at infinity.

For any Orlicz function Φ the statement Φ-*satisfies the suitable* Δ_2-*condition* means that:

Φ satisfies the Δ_2-condition for all t if μ is nonatomic and infinite.

Φ satisfies the Δ_2-condition at infinity if μ is nonatomic and finite.

Φ satisfies the Δ_2-condition at 0 if μ is a counting measure .

In the following , $L^0(\mu)$ will stand for the space of all (equivalence classes of) Σ-measurable real functions defined on Ω. For a given Orlicz function Φ we define, on $L^0(\mu)$, a convex functional (called a pseudomodular) by

$$I_\Phi(x) = \int_\Omega \Phi(x(t))d\mu.$$

The Orlicz space $L^\Phi(\mu)$ is defined to be the set of all $x \in L^0(\mu)$ such that $I_\Phi(\lambda x) < \infty$ for some $\lambda > 0$ depending on x. We endow $L^\Phi(\mu)$ with the Luxemburg norm

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(\frac{x}{\lambda}) \leq 1\}$$

and with the Orlicz norm

$$\|x\|_\Phi^0 = \sup\{\int_\Omega |x(t)y(t)|d\mu : y \in L^{\Phi^*}(\mu), I_{\Phi^*}(y) \leq 1\},$$

where the function $\Phi^*$is defined by the formula

$$\Phi^* = \sup\{uv - \Phi(v) : v \geq 0\}$$

and called complementary to Φ in the sense of Young.

It is well known that if Φ is finitely valued and satisfies the condition

$$\lim_{x\to\infty} \frac{\Phi(x)}{x} = 0,$$

then the Orlicz norm satisfies the Amemiya formula (see[**?**])

$$\|x\|_{\Phi}^{0} = \inf\{\frac{1}{k}(1 + \int_{\Omega} \Phi(kx)d\mu) : k > 0\}.$$

Moreover, there is a number k^* attaining the infimun, so that

$$\|x\|_{\Phi}^{0} = \frac{1}{k^*}(1 + \int_{\Omega} \Phi(k^* x)d\mu).$$

In [**?**], it is proved that Orlicz spaces generated by Orlicz functions satisfying the Δ_2-condition have the Kadec-Klee property for local convergence in measure. In this paper, we will study more deeply the relationship between the Δ_2-condition and the Kadec-Klee property for local convergence in measure or global convergence in measure for the Orlicz space $L^{\Phi}(\mu)$ with either the Luxemburg or the Orlicz norm.

In the sequel we will need some results concerning to order continuous Banach lattices. A Banach space E is said to be a Banach lattice if there exists in E a partial order, $\leq$, such that if $x, y \in E$ and $|x| \leq |y|$ then $\|x\| \leq \|y\|$. E is said to be order continuous if for all sequence $\{x_n\}$ in E^+ such that $x_n \searrow 0$ μ.a.e. we have $\|x_n\| \to 0$.

An easy proof of the following lemma, useful in the sequel, can be found in [**?**].

Lemma 1.1 *Let E be a Banach function lattice over a non-atomic σ-finite measure. If $x_n \to x$ then there exist $y \in E^+$, $(x_{n_k}) \subset (x_n)$ and $\varepsilon_{n_k} \subset R^+$ with $\varepsilon_{n_k} \downarrow 0$ sucht that $|x_{n_k} - x| \leq \varepsilon_{n_k} y$.*

2 Luxemburg norm

Our first result is the following

Theorem 2.1 *If E is a Banach function lattice and is not order continuous, then $E \notin H_l$.*

Proof: If E is not order continuous, it is well known, see[**?**] that there exist a sequence $\{x_n\}$ in E^+ with $\|x_n\| = 1$ and $\mathrm{supp} x_n \cap \mathrm{supp} x_m = \emptyset$ (which implies $x_n \to 0$ μ-a.e.) and a function $x \in E^+$ such that $x_n \leq x$ for any $n \in N$.

Define

$$y = \sum_{n=1}^{\infty} x_n \qquad \text{and} \qquad y_n = y - x_n.$$

If we could show that $y_n \to y$ weakly , or equivalently $x_n \to 0$ weakly, we would deduce $\|y_n\| \to \|y\|$ because $0 \leq y_n \leq y$.

However for any nonnegative $x^* \in E^*$ and for all $k \in N$ we have

$$\sum_{n=1}^{k} x^*(x_n) = x^*(\sum_{n=1}^{k} x_n) \leq x^*(x),$$

whence it follows that $\sum_{n=1}^{\infty} x^*(x_n)$ converges and so $x^*(x_n) \to 0$ as $n \to \infty$. Since every $x^* \in E^*$ can be written as a difference of two nonnegative functionals , we have shown that $x_n \to 0$ weakly. Therefore $\|y_n\| \to \|y\|$.

We also have that $y_n \to y$ μ-a.e. However

$$\|y - y_n\| = \|x_n\| = 1,$$

which means that $E \notin H_l$.

Corollary 2.2 *Let Φ be and arbitrary Orlicz function. Assume Φ does not satisfy the suitable Δ_2-condition. Then $L^{\Phi}(\mu) \notin H_l$.*

Proof:

The proof follows from the fact that the space $L^{\Phi}(\mu)$ is an order continuous Banach lattice if and only if Φ satisfies the suitable Δ_2-condition (see [**?**],[**?**] and[**?**]).

If μ is a finite measure the Kadec-Klee property for local and global convergence in measure are equivalent. So, we will restrict ourselves to study the case of an infinite measure.

Theorem 2.3 *Let Φ be and arbitrary Orlicz function , (Ω, Σ, μ) a nonatomic and infinite measure and $(L^{\Phi}(\mu), \|.\|_0)$ the Orlicz space endowed with the Luxemburg norm. Assume $a_{\Phi} > 0$ and Φ satisfies the Δ_2-condition at ∞. Then $L^{\Phi}(\mu) \notin H_g$.*

Proof: Consider a sequence of measurable set $\{A_n\}$ such that

$$\mu(A_n) = 2^{-n}.$$

Set $A = \cup A_n$.

Define

$$x = a_{\Phi}\chi_{\Omega \setminus A} \quad \text{and} \quad x_n = a_{\Phi}\chi_{\Omega \setminus A} + b_n \chi_{A_n}$$

where $1 = \Phi(b_n)\mu(A_n)$. Such a sequence (b_n) exists since Φ satisfies the Δ_2 condition at ∞.

We first note that $x_n - x = b_n \chi_{A_n}$. Therefore $x_n \to x$ globally in measure.

Now we are going to show that

$$\|x\|_{\Phi} = \|x_n\|_{\Phi} = 1.$$

We have

$$I_{\Phi}(x) \leq I_{\Phi}(x_n) = \Phi(a_{\Phi})\mu(\Omega \setminus A) + \Phi(b_n)\mu(A_n) = 1.$$

Since Φ satisfies the Δ_2-condition at ∞ this implies (see [**?**])

$$\|x\|_\Phi \leq \|x_n\|_\Phi = 1. \tag{1}$$

On the other hand for all $\lambda > 1$,

$$I_\Phi(\lambda x) = \Phi(\lambda a_\Phi)\mu(\Omega \setminus A) = +\infty.$$

So $\|\lambda x\|_\Phi \geq 1$ which implies $\|x\|_\Phi \geq 1$. Hence by (1) using we obtain

$$\|x\|_\Phi = \|x_n\|_\Phi = 1$$

In order to finish the proof we only need to prove that $\|x - x_n\|_\Phi$ does not converge to 0. But

$$I_\Phi(x_n - x) = \Phi(b_n)\mu(A_n) = 1$$

which implies $\|x_n - x\|_\Phi = 1$.

Theorem 2.4 *Let Φ be and arbitrary Orlicz function, (Ω, Σ, μ) a nonatomic and infinite measure and $(L^\Phi(\mu), \|.\|)_\Phi)$ the Orlicz space endowed with the Luxemburg norm. Assume Φ does not satisfy the Δ_2-condition at 0, Φ vanishes only at zero and Φ satisfies the Δ_2-condition at ∞. Then $L^\Phi(\mu) \notin H_g$.*

Proof: Assume Φ does not satisfy the Δ_2-condition at 0. Then, there exists a sequence (u_n) of positive real numbers with $u_n \to 0$ and such that

$$\Phi((1 + \frac{1}{n})u_n) > 2^n \Phi(u_n).$$

Divide Ω into two disjoint parts A and B such that $\mu(A) = +\infty$ and $\mu(B) > 0$.

Select a sequence (A_n) of pairwise disjoint measurable subsets of A such that

$$\Phi(u_n)\mu(A_n) = 2^{-n-1}.$$

Select a sequence (B_n), $B_n \in \Sigma\,, B_n \subset B$ such that $\mu(B_n) \to 0$.

Consider $b_n > 0$ satisfying

$$\Phi(b_n)\mu(B_n) = \frac{1}{2}$$

Define

$$x_n = \sum_{k=1}^{\infty} u_{2k-1}\chi_{A_{2k-1}} + b_{2n}\chi_{B_{2n}} \text{ and } x = \sum_{k=1}^{\infty} u_{2k-1}\chi_{A_{2k-1}}.$$

We first note that

$$x_n - x = b_{2n}\chi_{B_{2n}} \to 0$$

globally in measure.

Given $\lambda > 1$ there exists $n_0 \in N$ such that $(1 + \frac{1}{n}) < \lambda$ for all $n > n_0$. Thus

$$I_\Phi(\lambda x) = \sum_{k=1}^{\infty} \Phi(\lambda u_{2k-1})\mu(A_{2k-1}) \geq \sum_{2k-1>n_0} \Phi((1+\frac{1}{2k-1})u_{2k-1})\mu(A_{2k-1}) = +\infty.$$

So

$$\|x\|_\Phi \geq 1 \tag{2}$$

We observe now that

$$I_\Phi(x) \leq I_\Phi(x_n) = \sum_{k=1}^{\infty} \Phi(u_{2k-1})\mu(A_{2k-1}) + \Phi(b_{2n})\mu(B_{2n}) = 1,$$

which with (2) give us

$$\|x_n\|_\Phi = \|x\|_\Phi = 1$$

In order to finish the proof we only need to prove that $\|x - x_n\|_\Phi$ does not converge to 0. But

$$I_\Phi(x_n - x) = \Phi(b_{2n})\mu(B_{2n}) = \frac{1}{2}.$$

Since Φ satisfies the Δ_2-condition at ∞, there exists $\delta > 0$ such that

$$\|x_n - x\|_\Phi > \delta > 0.$$

Theorem 2.5 *Let Φ be and arbitrary Orlicz function , (Ω, Σ, μ) a nonatomic and infinite measure and $(L^\Phi(\mu), \|.\|_\Phi)$ the Orlicz space endowed with the Luxemburg norm. Assume Φ does not satisfy the Δ_2-condition at ∞. Then $L^\Phi(\mu) \notin H_g$.*

Proof:

If we assume that Φ does not satisfy the Δ_2-condition at infinity then for all $K \in R^+$ there exists $u_{n(K)} \geq n$ such that

$$\Phi((1 + \frac{1}{n})u_{n(K)}) > K\Phi(u_{n(K)}).$$

Consider $K = 2^{n+1}$. There exist $u_n \geq n$ such that

$$\Phi((1 + \frac{1}{n})u_n) > 2^{n+1}\Phi(u_n).$$

Let $A_n \in \Sigma$ be such that

$$\Phi(u_n)\mu(A_n) = 2^{-n}.$$

Define

$$x = \sum_{k=1}^{\infty} u_k \chi_{A_k} \text{ and } x_n = \sum_{k \neq n} u_k \chi_{A_k}$$

We first note that

$$x_n - x = u_n \chi_{A_n} \to 0$$

globally in measure.

We have

$$I_\Phi(x_n) \leq I_\Phi(x) = \sum_{n=1}^{\infty} \Phi(u_n)\mu(A_n) = 1.$$

So $\|x_n\|_\Phi \leq \|x\|_\Phi \leq 1$.

On the other hand, taking any $\lambda > 1$ it can be proved that

$$I_\Phi(\lambda x_n) = +\infty$$

Thus $\|x_n\|_\Phi = \|x\|_\Phi = 1$. To finish the proof it suffices to show that (x_n) does not converge to x in $(L^\phi(\mu), \|.\|_0)$. However

$$I_\Phi((1 + \frac{1}{n})(x_n - x)) = \Phi((1 + \frac{1}{n})u_n)\mu(A_n) > 2^n \Phi((1 + \frac{1}{n}u_n)\mu(A_n) = 1.$$

Thus $\|x_n - x\|_\Phi \geq (1 + \frac{1}{n})$

All the previous results can be summarize in the following theorem

Theorem 2.6 *Let Φ and arbitrary Orlicz function and $(L^\Phi(\mu), \|\cdot\|_\Phi)$ the Orlicz space endowed with the Luxemburg norm. The following statements are equivalent*

1. *Φ satisfies the suitable Δ_2-condition.*
2. *$L^\Phi(\mu) \in H_g$*
3. *$L^\Phi(\mu) \in H_l$*

3 Orlicz Norm

As usual, L^1 denotes the Lebesgue space of these x in L^0 that

$$\|x\|_1 = \int_{R^+} |x(t)|dt < \infty$$

and L^∞ denotes the space of m-essentially bounded functions in L^0 equipped with the norm

$$\|x\| = \operatorname{ess\,sup} |x(t)|.$$

Consider the following norms for $L^1 \cap L^\infty$ and $L^1 + L^\infty$

$$\|x\|_{L^1 \cap L^\infty} = \sup(\|x\|_1, \|x\|_\infty)$$

and

$$\|x\|_{L^1+L^\infty} = \inf(\|u\|_1 + \|v\|_\infty),$$

respectively, where the supremum and the infimum are taken over all $u \in L^1$, $v \in L^\infty$ such that $u + v = x$.

In [?] ,[?] it is proved that $L^1 \cap L^\infty = L^\psi$ and $L^1 + L^\infty = L^\phi$, where $\psi(u) = |u|$ for $|u| \leq 1$, $\psi(u) = \infty$ for $|u| > 1$ and $\phi(u) = \max(0, |u| - 1)$. The functions ψ and ϕ are mutually complemented in the sense of Young.

It is well known that $\|x\|_\psi = \|x\|_{L^1 \cap L^\infty}$ for any $x \in L^\psi$ (see[?]). Moreover, the spaces

$$(L^1 \cap L^\infty, \|x\|_{L^1 \cap L^\infty}), (L^1 + L^\infty, \|x\|_{L^1+L^\infty}) \text{ and } (L^\psi, \|.\|_\psi), (L^\phi, \|.\|_\phi^0)$$

form two couples of dual spaces in the Kothe's sense. Therefore

$$\|.\|_\phi^0 = \|x\|_{L^1+L^\infty}$$

for all $x \in L^\phi$. Additionally, in [?] it is proved the Amemiya formula for the norm in $L^1 + L^\infty$.

For any $x \in L^0$ the decreasing rearrangement of x is the function x^* defined by

$$x^*(t) = \inf\{\lambda > 0 : d_x(t) < \lambda\}$$

where $d_x(t)$ is the distribution function defined by

$$d_x(t) = \mu(\{w \in \Omega : |x(w)| > t\}$$

For our purposes, it is worthwhile to note, see[?], that

$$(x_1 + x_2)^*(t_1 + t_2) \leq x_1^*(t_1) + x_2^*(t_2)$$

and for all $x \in L^1 + L^\infty$ the equality

$$\|x\|_{L^1+L^\infty} = \int_0^1 x^*(t)dt$$

holds.

From Theorem 2.5 we know that $(L^\Phi(\mu), \|\cdot\|_\Phi)$ does not satisfy the Kadec-Klee property for global convergence in measure if Φ does not satisfy the Δ_2-condition at ∞. However, this fact is not true when the Orlicz norm is considered.

Theorem 3.1 *The space $(L^1+L^\infty, \|x\|_{L^1+L^\infty})$ satifies the Kadec-Klee propertie for global convergence in measure ($L^1 + L^\infty \in H_g$).*

Proof: Assume $(x_n) \subset L^1 + L^\infty$, $x \in L^1 + L^\infty$, $x_n \to x$ globally in measure and $\|x_n\|_{L^1+L^\infty} = \|x\|_{L^1+L^\infty} = 1$.

Since $x_n \to x$ globally in measure we have $x_n^* \to x^*$ μ-a.e. Thus

$$x_n^* \chi_{[0,1]} \to x^* \chi_{[0,1]} \mu - \text{a.e.}$$

Bearing in mind that $L^1 \in H_l$ and $\|x_n^* \chi_{[0,1]}\|_{L^1} = \|x^* \chi_{[0,1]}\|_{L^1} = 1$ we deduce

$$\int_0^1 |x_n^*(s) - x^*(s)| ds \to 0.$$

By Lemma 1.1 there exists $(x_{n_k}^*)$ a subsequence of x_n^* and $y \geq 0$, $y \in L^1[0,1]$ such that $|x_{n_k}^*(t) - x^*(t)| \leq y(t)$a.e. in $[0,1]$.

On the other hand

$$(x_{n_k} - x)^*(t) \leq x_{n_k}^*(\frac{t}{2}) + x^*(\frac{t}{2}) \leq 2x^*(\frac{t}{2}) + y(\frac{t}{2}).$$

Therefore, by applying the Lebesgue dominated convergence theorem we obtain

$$\int_0^1 (x_{n_k} - x)^*(t) dt \to 0$$

which is equivalent to

$$\|x_{n_k} - x\|_{L^1 + L^\infty} \to 0.$$

Thus, since for each subsequence of $(x_n - x)$ we we can extract a subsequence which converges in norm to 0, we have

$$\|x_n - x\|_{L^1 + L^\infty} \to 0$$

and the proof is concluded.

Our last result is the following

Theorem 3.2 *If Φ does not satisfy the Δ_2-condition at ∞ and μ is nonatomic then $(L^\Phi(\mu), \|\cdot\|_\Phi^0) \notin H_g$.*

Proof: If μ is finite it is obvious , because in this case se we have

$$H_g \Leftrightarrow H_f \Rightarrow O.C. \Rightarrow \Delta_2(\infty).$$

Assume that μ is nonatomic and infinite and $\Phi \notin \Delta_2(\infty)$. There exists a sequence (u_n) of nonnegative real numbers such that $u_n \uparrow \infty$ and

$$\Phi(2u_n) > 2^n \Phi(u_n).$$

Take any nonnegative $x \in L^\Phi$ with $\|x\|_\Phi^0 = 1$. Since Φ does not satisfies the Δ_2-condition at ∞ we have $\lim_{t\to\infty} \frac{\Phi(t)}{t} = 0$ and so the Amemiya formula is satisfied.

There exists a sequence (A_n) in Σ with $\mu(A_n) = \infty$ and such that

$$I_\Phi(2kx\chi_{A_n}) \leq 2^{-n},$$

where $k \geq 1$ satisfies

$$\|x\|_\Phi^0 = \frac{1}{k}(1 + I_\Phi(kx)).$$

Let $B_n \subset A_n$ be such that

$$\Phi(u_n)\mu(B_n) = 2^{-n}$$

and define

$$x_n = x + \frac{u_n}{2k}\chi_{B_n} = x\chi_{\Omega \setminus B_n} + (x + \frac{u_n}{2k})\chi_{B_n}.$$

Since $x_n \geq x \geq 0$, we have $\|x_n\|_\Phi^0 \geq \|x\|_\Phi^0 = 1$.

On the other hand

$$\begin{aligned}
\|x_n\|_\Phi^0 &= \inf_{\rho > 0} \frac{1}{\rho}(1 + I_\Phi(\rho x_n)) \leq \\
&\leq \frac{1}{k}(1 + I_\Phi(kx_n)) = \\
&= \frac{1}{k}(1 + I_\Phi(kx\chi_{\Omega \setminus B_n}) + I_\Phi(kx\chi_{B_n} + \frac{u_n}{2}\chi_{B_n}) \leq \\
&\leq 1 + \frac{1}{2}(I_\Phi(2kx\chi_{B_n}) + I_\Phi(u_n\chi_{B_n})) \leq \\
&\leq 1 + \frac{1}{2}(\frac{1}{2^n} + \frac{1}{2^n}) \to 1
\end{aligned}$$

So $\|x_n\|_\Phi^0 \to \|x\|_\Phi^0$. Furthemore $x_n \to x$ globally in measure.

We are going to show that $\|x_n - x\|_\Phi^0 \geq \frac{1}{4}$ and the proof will be concluded. We have

$$I_\Phi(4k(x_n - x)) = I_\Phi(2u_n\chi_{B_n}) = \Phi(2u_n)\mu(B_n)) > 1.$$

So $\|x_n - x\|_\Phi \geq \frac{1}{4k}$ and since $\|x_n - x\|_\Phi^0 \geq \|x_n - x\|_\Phi$ the proof is concluded.

For the case of the Orlicz norm we summarize our results in the following theorem.

Theorem 3.3 *Let Φ and arbitrary Orlicz function, $(L^\Phi(\mu), \|\cdot\|_\Phi^0)$ the Orlicz space endowed with the Orlicz norm and μ nonatomic. Then*

1. *$\Phi \notin \Delta_2(\infty) \Rightarrow L^\Phi \notin H_g$.*
2. *Φ needs not to vanish only at zero to have property H_g ($L^1 + L^\infty$ as an example).*

We finish with an open question: Assume that Φ vanishes outside 0 and $\Phi \notin \Delta_2(\infty)$. Is it true that $(L^\Phi(\mu), \|\cdot\|_\Phi^0) \notin H_g$.

References

[1] V.I. Chillin, P.G.Dodsds, A.A. Sedaev and F.A.Sukochev, *Characterization of Kadec-Klee properties in symmetric spaces of measurable functions*, Trans. Amer. Math. Soc. **Vol. 348**, (1996), 4895-4918.

[2] H. Hudzik , *On some equivalent conditions in Musielaks- Orlicz spaces*, Comment. Math. Prace Mat. **24**, (1984), 57-64.

[3] H. Hudzik , *Intersections and algebraic sums of Musielaks- Orlicz spaces*, Portugaliae Math. 40**Vol. 3**, (1993), 287-296.

[4] H. Hudzik , *On smallest and largest Orlicz spaces*, Math. Nachr. 141, (1989), 109-115.

[5] H. Hudzik , *On the distance from the subspace of order continuous elements in* $L^1 + L^\infty$, Functiones et Approximatio XXV, (1997), 157-163.

[6] H. Hudzik and M. Mastylo, *strongly extreme points in kothe-Bochner spaces*, Rocky Mountain J. Math. 3**Vol. 23**, (1993), 899-909.

[7] A. Kaminska , *Flat Orlicz- Musielak sequence spaces*, Bull. Acad. Polo. Sci. Math. 30**Vol. 30**, (1982), 347-352.

[8] L.V. Kantarovic and G.P. Akilov, *Functional Analysis*, 2nd edition, Moscow (1978), in Russian.

[9] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II*, Springe-Verlag, Berlin (1979).

[10] A. Medzhitov and P. Sukochev, *The property (H) in Orlicz spaces* , Bull. Polish. Acad. Sci. Math. **40** (1992), 5-11.

[11] M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces* , Pure and Applied Mathematics, Marcel Dekker. New-York,1991

[12] F. Riesz, *Sur la convergence en moyenne I* , Acta Sci. Math. **4** 1928/29, 58-64.

[13] F. Riesz, *Sur la convergence en moyenne II* , Acta Sci. Math. **4** 1928/29, 182-185.

[14] B. Turett , *Rotundity of Orlicz spaces*, Indag. Math. **38**, (1976), 462-469.