# ASYMPTOTICALLY REGULAR MAPPINGS IN MODULAR FUNCTION SPACES

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## 1. ABSTRACT

Let $\rho$ be a modular function satisfying a $\Delta_2$-type condition and $L_\rho$ the corresponding modular space. The main result in this paper states that if $C$ is a $\rho$-bounded and $\rho$-a.e sequentially compact subset of $L_\rho$ and $T : C \to C$ is an asymptotically regular mapping such that $\liminf_{n\to\infty} |T^n| < 2$, where $|S|$ denotes the Lipschitz constant of $S$, then $T$ has a fixed point. We show that the estimate $\liminf_{n\to\infty} |T^n| < 2$ cannot be, in general, improved.

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## 2. INTRODUCTION

Let $X$ be a metric space. A mapping $T : X \to X$ is said to be asymptotically regular if $\lim_n d(T^{n+1}x, T^n x) = 0$ for each $x \in X$. This notion was defined by Browder and Petryshyn [2]. The existence of fixed points for asymptotically regular mappings has been widely studied [3, 4, 5, 6, 7, 8, 9, 10, 11, 16, 17]. When $X$ is a convex, closed, subset of a Banach space it is known [12] that the problem of the existence of fixed point for a nonexpansive mapping is equivalent to the same problem for a nonexpansive asymptotically regular mapping. On the other hand, the theory of modular spaces was initiated by Nakano [21] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [20] in 1959. Besides the idea of defining a norm and considering particular Banach spaces of functions, another direction is based on considering an abstractly given funtional defined on a linear space of functions which controls the growth of members of the space. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated in modular spaces (see, for instance [13] and references therein). In this paper, we study the existence of fixed points for asymptotically regular mapping defined from a $\rho$-bounded and $\rho$-a.e sequentially compact set $C$ of a modular space $L_\rho$ into $C$. We actually prove that $T$ has a fixed point if $\liminf_n |T^n| < 2$, $|T|$ being the exact Lipschitz constant of $T$. It is worthed to notice the simplicity of this statement in comparison with similar results in Banach spaces, where a different upper bound for $\liminf_n |T^n|$ must be considered for each space. Even in Hilbert spaces (see [1, chapter IX]) a best estimate is unknown. We also give an example showing that this result can not be, in general, improved.

## 3. PRELIMINARIES

We start by recalling some basic concepts and facts of modular spaces as formulated by Kozlowski. For more details the reader is refered to [13], [14], [15] and [19].

Let $\Omega$ be a nonempty set and $\Sigma$ be a nontrivial $\sigma$-algebra of subsets of $\Omega$. Let $\mathcal{P}$ be a $\delta$-ring of subsets of $\Sigma$, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that

$\Omega = \bigcup K_n$. In other words, the family $\mathcal{P}$ plays the role of the $\delta$-ring of subsets of finite measure. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}$ we will denote the space of all measurable functions, i.e. all functions $f : \Omega \to \Re$ such that there exists a sequence $\{g_n\} \in \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By $1_A$ we denote the characteristic function of the set $A$.

**Definition 3.1.** A functional $\rho : \mathcal{E} \times \Sigma \to [0, \infty]$ is called a function modular if

- $(P_1)$ $\rho(0, E) = 0$ for any $E \in \Sigma$,
- $(P_2)$ $\rho(f, E) \leq \rho(g, E)$ whenever $|f(\omega)| \leq |g(\omega)|$ for any $\omega \in \Omega$, $f, g \in \mathcal{E}$ and $E \in \Sigma$,
- $(P_3)$ $\rho(f, .) : \Sigma \to [0, \infty]$ is a $\sigma$-subadditive measure for every $f \in \mathcal{E}$,
- $(P_4)$ $\rho(\alpha, A) \to 0$ as $\alpha$ decreases to 0 for every $A \in \mathcal{P}$, where $\rho(\alpha, A) = \rho(\alpha 1_A, A)$,
- $(P_5)$ if there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$ for every $\beta > 0$,
- $(P_6)$ for any $\alpha > 0$ $\rho(\alpha, .)$ is order continuous on $\mathcal{P}$, that is $\rho(\alpha, A_n) \to 0$ if $\{A_n\} \in \mathcal{P}$ and decreases to $\emptyset$.

The definition of $\rho$ is then extended to $f \in \mathcal{M}$ by

$$\rho(f, E) = \sup\{\rho(g, E); g \in \mathcal{E},\ |g(\omega)| \leq |f(\omega)|\ \ \omega \in \Omega\}.$$

This will enable us to define $\rho(\alpha, E)$ for sets $E$ not in $\mathcal{P}$; for the sake of simplicity, we write $\rho(f)$ instead of $\rho(f, \Omega)$.

**Definition 3.2.** A set $E$ is said to be $\rho$-null if and only if $\rho(\alpha, E) = 0$ for $\alpha > 0$. A property $p(\omega)$ is said to hold $\rho$-almost everywhere ($\rho$-a.e.) if the set $\{\omega \in \Omega;\ p(\omega)\ \text{ does not hold }\}$ is $\rho$-null. For example we will say frequently $f_n \to f$ $\rho$-a.e.

Note that a countable union of $\rho$-null sets is still $\rho$-null. In the sequel we will identify sets $A$ and $B$ whose symmetric difference $A \Delta B$ is $\rho$-null; similarly we will identify measurable functions which differ only on a $\rho$-null set.

It is easy to see that the functional $\rho : \mathcal{M} \to [0, \infty]$ is a modular because it satisfies the following properties:

(i) $\rho(f) = 0$ iff $f = 0$ $\rho$-a.e.
(ii) $\rho(\alpha\, f) = \rho(f)$ for every scalar $\alpha$ with $|\alpha| = 1$ and $f \in \mathcal{M}$.
(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha \geq 0, \beta \geq 0$ and $f, g \in \mathcal{M}$.

In addition, if the following property is satisfied

(iii)' $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$ ; $\alpha \geq 0, \beta \geq 0$ and $f, g \in \mathcal{M}$,

we say that $\rho$ is a convex modular. The modular $\rho$ defines a corresponding modular space, i.e the vector space $L_\rho$ given by

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

The modular space $L_\rho$ can be equipped with an $F$-norm defined by

$$||f||_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{f}{\alpha}\right) \leq \alpha\right\}.$$

When $\rho$ is convex the formula

$$||f||_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{f}{\alpha}\right) \leq 1\right\}$$

**Definition 3.3.**

(a) The sequence $\{f_n\} \subset L_\rho$ is said to be $\rho$-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$,
(a') The sequence $\{f_n\} \subset L_\rho$ is said to be $\rho$-a.e convergent to $f \in L_\rho$ if the set $\{\omega \in \Omega; f_n(\omega) \not\to f(\omega)\}$ is $\rho$-null.
(b) The sequence $\{f_n\} \subset L_\rho$ is said to be $\rho$-Cauchy if $\rho(f_n - f_m) \to 0$ as $n$ and $m$ go to $\infty$,
(b') The sequence $\{f_n\} \subset L_\rho$ is said to be $\rho$-a.e Cauchy if $\{\omega \in \Omega; \{f_n(\omega)\}$ is not a Cauchy sequence $\}$ is $\rho$-null.
(c) A subset $C$ of $L_\rho$ is called $\rho$-closed if the $\rho$-limit of a $\rho$-convergent sequence of $C$ always belongs to C.
(c') A subset $C$ of $L_\rho$ is called $\rho$-a.e sequentially closed if the $\rho$-a.e limit of a $\rho$-a.e convergent sequence of $C$ always belongs to C.

(d) A subset $C$ of $L_\rho$ is called $\rho$-sequentially compact if every sequence in $C$ has a $\rho$-convergent subsequence in $C$.

(d') A subset $C$ of $L_\rho$ is called $\rho$-a.e sequentially compact if every sequence in $C$ has a $\rho$-a.e convergent subsequence in $C$.

(e) A subset $C$ of $L_\rho$ is called $\rho$-bounded if

$$\delta_\rho(C) = \sup\{\rho(f-g); f, g \in C\} < \infty.$$

**Definition 3.4.** Let $\rho$ be a function modular, we define a growth funtion $\omega_\rho : [0,\infty] \to [0,\infty]$ by

$$\omega_\rho(t) = \sup\left\{\frac{\rho(tf)}{\rho(f)}; f \in L_\rho, 0 < \rho(f) < \infty\right\}, t \geq 0.$$

The following technical result [13] is fundamental for this work.

**Lemma 3.1.** Let $\{f_n\}_n$ be a sequence in $E_\rho$ such that $f_n \overset{\rho-a.e}{\to} f \in E_\rho$ and there exists $k > 1$ such that $\sup_n \rho(k(f_n - f)) < \infty$. Then, we have

$$\liminf_{n\to\infty} \rho(f_n - g) = \liminf_{n\to\infty} \rho(f_n - f) + \rho(f-g) \quad \text{for all} \;\; g \in E_\rho$$

and, therefore,

$$\liminf_{n\to\infty} \rho(f_n - f) \leq \liminf_{n\to\infty} \rho(f_n - g) \;\; \text{for all} \;\; g \in E_\rho.$$

¿From this lemma a uniform Opial property-type for $E_\rho$ can be derived. Recall [22] that a Banach space is said to satisfy the uniform Opial property with respect to an arbitrary topology $\tau$ if for every $c > 0$ there exists $r > 0$ such that $\liminf_{n\to\infty} ||x_n + x|| \geq 1 + r$, if $\{x_n\}_n$ is a $\tau$-null sequence with $\liminf_{n\to\infty} ||x_n|| \geq 1$ and $||x|| \geq c$. From lemma 3.1 it is clear that $\liminf_{n\to\infty} \rho(f_n + f) \geq 1 + c$ if $\rho(f) \geq c$ and $\{f_n\}_n$ is a $\rho$-a.e null sequence which satisfies $\liminf_{n\to\infty} \rho(f_n) \geq 1$ and there exists $K > 1$ such that $\sup_n \rho\big(K(f_n - f)\big) < \infty$.

## 4. SOME TECHNICAL RESULTS

**Definition 4.1.** Let $\rho$ be a function modular. We say that $\rho$ satisfies the $\Delta_2$-type condition if there exists $K > 0$ such that $\rho(2f) \leq K\rho(f)$ for all $f \in L_\rho$.

As examples of convex funtion modular with $\Delta_2$-type condition we mention, the usual $l^p$ spaces and function modular of Orlicz spaces, where the measure space $(\Omega, \Sigma, \mu)$ is $\sigma$-finite, the measure $\mu$ is atomless and infinite, and the Orlicz function $\psi$ is convex satisfying $\Delta_2$-type condition, i.e.

$$\limsup_{u\to\infty} \frac{\psi(2u)}{\psi(u)} < \infty \text{ and } \limsup_{u\to 0} \frac{\psi(2u)}{\psi(u)} < \infty.$$

Note that the $\Delta_2$-type condition implies the $\Delta_2$-condition. For this condition we refer to [15] and [19]. In the sequel, we will assume that $\rho$ is convex and satisfies the $\Delta_2$-condition. In this case we have $L_\rho = E_\rho$.
The following lemma can be easily proved.

**Lemma 4.1.** The growth funtion $\omega_\rho$ has the following properties:

(1) $\omega_\rho(t) < \infty, \forall t \in [0, \infty)$

(2) $\omega_\rho : [0, \infty) \to [0, \infty)$ is a convex, strictly increasing function. So, it is continuous.

(3) $\omega_\rho(\alpha\beta) \leq \omega_\rho(\alpha)\omega_\rho(\beta); \forall \alpha, \beta \in [0, \infty)$

(4) $\omega_\rho^{-1}(\alpha)\omega_\rho^{-1}(\beta) \leq \omega_\rho^{-1}(\alpha\beta); \forall \alpha, \beta \in [0, \infty)$, where $\omega_\rho^{-1}$ is the inverse function of $\omega_\rho$.

The following is a technical lemma which will be needed because of lack of the triangular inequality.

**Lemma 4.2.** Let $\{f_n\}$ and $\{g_n\}$ be two sequences in $L_\rho$. Then

$$\lim_{n\to\infty} \rho(g_n) = 0 \Longrightarrow \limsup_{n\to\infty} \rho(f_n + g_n) = \limsup_{n\to\infty} \rho(f_n).$$

*Proof.* By property (iii) of the modular $\rho$, we have

$$\rho(f_n + g_n) \leq \rho\left(\frac{f_n}{1-\varepsilon}\right) + \rho\left(\frac{g_n}{\varepsilon}\right), \forall \varepsilon \in (0,1)$$

Thus,

$$\rho(f_n + g_n) \leq \omega\left(\frac{1}{1-\varepsilon}\right)\rho(f_n) + \omega\left(\frac{1}{\varepsilon}\right)\rho(g_n)$$

and

$$\limsup_{n\to\infty} \rho(f_n + g_n) \leq \omega\left(\frac{1}{1-\varepsilon}\right) \limsup_{n\to\infty} \rho(f_n)$$

Since $\varepsilon$ is arbitrary and

$$\omega\left(\frac{1}{1-\varepsilon}\right) \to 1 \text{ as } \varepsilon \to 0^+$$

we obtain

$$\limsup_{n\to\infty} \rho(f_n + g_n) \leq \limsup_{n\to\infty} \rho(f_n).$$

Furthermore, the same argument proves

$$\begin{aligned} \limsup_{n\to\infty} \rho(f_n) &= \limsup_{n\to\infty} \rho(f_n + g_n - g_n) \\ &\leq \limsup_{n\to\infty} \rho(f_n + g_n). \end{aligned}$$

The following lemma shows that the growth function can be used to give an upper bound for the norm of a function.

**Lemma 4.3.** Let $L_\rho$ be a function modular space satisfying the $\Delta_2$-type condition. Then

$$||f||_\rho \leq \frac{1}{\omega^{-1}\left(\frac{1}{\rho(f)}\right)}$$

*Proof.* Assume, $\alpha < ||f||_\rho$. We have $1 < \rho(f/\alpha)$ which implies

$$\frac{1}{\rho(f)} < \omega\left(\frac{1}{\alpha}\right)$$

and so

$$\omega^{-1}\left(\frac{1}{\rho(f)}\right) < \frac{1}{\alpha}.$$

Letting $\alpha \to ||f||_\rho^{\,-}$, we obtain $||f||_\rho \leq \dfrac{1}{\omega^{-1}\left(\frac{1}{\rho(f)}\right)}$.

## 5. MAIN RESULT

In this section we present the main result of this work. Indeed, we will prove a fixed point theorem for asymptotically regular mappings. Certainly, one can also consider mappings which are asymptotically regular with respect to the F-norm induced by the modular function. We should like to mention that, generally speaking, there is no natural relation between these two kinds of asymptotically regularness. Indeed all results expressed in terms of modulars are more convenient in the sense that their assumptions are much easier to verify.

Let $C$ a subset of $L_\rho$ and $T : C \to C$, we denote by $|T|$ the exact Lipschitz constant of $T$, i.e.

$$|T| = \sup\left\{\frac{\rho(Tf - Tg)}{\rho(f-g)} : f \neq g,\ f, g \in C\right\} \text{ and } s(T) = \liminf_{n\to\infty} |T^n|.$$

**Theorem 5.1.** Let $\rho$ be a convex modular function satisfying the $\Delta_2$-type condition, $C$ a $\rho$-bounded, $\rho - a.e$ sequentially compact subset of $L_\rho$. Let $T : C \to C$ be an asymptotically regular mapping such that $s(T) < 2$. Then, T has a fixed point.

*Proof.* Choose a sequence $\{n_k\}$ of positive integers such that $s(T) = \lim_{k\to\infty} |T^{n_k}| = \liminf_{n\to\infty} |T^n|$ and define a function $r$ on C by

$$r(f) = \inf\{r > 0 : \exists g \in C \text{ such that } \liminf_{k\to\infty} \rho(f - T^{n_k} g) \leq r\}.$$

Since $s(T) < 2$, there exists $b \in (1, 2)$ such that $s(T) < b < 2$. Let $\varepsilon \in (0, 1)$ such that $\omega_\rho\left(\dfrac{1}{\varepsilon}\right) < \dfrac{1}{b-1}$ and choose $\gamma \in (0, 1)$ such that $\omega_\rho\left(\dfrac{1}{\varepsilon}\right) < \dfrac{\gamma}{b-1}$. Since $\gamma + (1-b)\omega_\rho\left(\frac{1}{\varepsilon}\right) > 0$, we can choose $\delta \in (0, 1)$ such that

$\delta < \gamma + (1-b)\omega_\rho\left(\frac{1}{\varepsilon}\right)$ . Now, we choose a number $\mu \in (0,1)$ such that

$$\mu < \min\left\{\frac{\delta}{\omega_\rho\left(\frac{1}{1-\varepsilon}\right)}\ ,\ \frac{\gamma - \delta + \omega_\rho\left(\frac{1}{\varepsilon}\right)(1-b)}{\omega_\rho\left(\frac{1}{\varepsilon}\right)b}\right\}$$

Finally,denote

$$\alpha = \max\left\{1 + \mu - \frac{\delta}{\omega_\rho\left(\frac{1}{1-\varepsilon}\right)}\ ,\ b(1+\mu) - \frac{\gamma-\delta}{\omega_\rho\left(\frac{1}{\varepsilon}\right)}\right\}.$$

Then $0 < \alpha < 1$. Since $\gamma < 1$, we can find a positive integer $k_0$ such that $|T^{n_{k_0}}| < b$ and

$$\rho(f - T^{n_{k_0}} f) > \gamma\, r(f).$$

Since $\mu > 0$, we can also find $g \in C$ such that

$$\liminf_{k\to\infty} \rho(f - T^{n_k} g) \leq r(f)(1+\mu).$$

Consider a subsequence $\{n_{k'}\}$ of $\{n_k\}$ such that $\{T^{n_{k'}} g\}$ is $\rho$-a.e convergent in $C$, say to $h$, and

$$\lim_{k'\to\infty} \rho(f - T^{n_{k'}} g) = \liminf_{k\to\infty} \rho(f - T^{n_k} g).$$

Therefore, using Lemma 4.2 and the asymptotic regularity of $T$ we have

$$\begin{aligned}
\limsup_{k'\to\infty} \rho(T^{n_{k_0}} f - T^{n_{k'}} g) &\leq |T^{n_{k_0}}| \limsup_{k'\to\infty} \rho(f - T^{n_{k'} - n_{k_0}} g) \\
&= |T^{n_{k_0}}| \limsup_{k'\to\infty} \rho(f - T^{n_{k'}} g + T^{n_{k'}} g - T^{n_{k'} - n_{k_0}} g) \\
&= |T^{n_{k_0}}| \limsup_{k'\to\infty} \rho(f - T^{n_{k'}} g).
\end{aligned}$$

We split the proof into two cases:

**Case 1.** Assume that

$$\rho(f - h) \geq \frac{\delta}{\omega_\rho\left(\frac{1}{1-\varepsilon}\right)} r(f)$$

Using lemma 3.1, we have

$$\begin{aligned}\liminf_{k'\to\infty}\rho(T^{n_{k'}}g-h) &= \liminf_{k'\to\infty}\rho(T^{n_{k'}}g-f)-\rho(f-h)\\ &\le (1+\mu)r(f)-\frac{\delta}{\omega_\rho\left(\frac{1}{1-\varepsilon}\right)}r(f)\\ &= \left[(1+\mu)-\frac{\delta}{\omega_\rho\left(\frac{1}{1-\varepsilon}\right)}\right]r(f)\\ &\le \alpha\, r(f).\end{aligned}$$

**Case 2.** Assume that

$$\rho(f-h)<\frac{\delta}{\omega_\rho\left(\frac{1}{1-\varepsilon}\right)}r(f)$$

Applying again Lemma 3.1, we have

$$\liminf_{k'\to\infty}\rho(T^{n_{k'}}g-h)=\liminf_{k'\to\infty}\rho(T^{n_{k'}}g-T^{n_{k_0}}f)-\rho(T^{n_{k_0}}f-h).$$

The property (iii) of the modular and the definition of the growth function $\omega_\rho$ give

$$\rho(T^{n_{k_0}}f-f)\le\omega_\rho\left(\frac{1}{\varepsilon}\right)\rho(T^{n_{k_0}}f-h)+\omega_\rho\left(\frac{1}{1-\varepsilon}\right)\rho(h-f)$$

then

$$\rho(T^{n_{k_0}}f-h)\ge\frac{1}{\omega_\rho\left(\frac{1}{\varepsilon}\right)}\rho(T^{n_{k_0}}f-f)-\frac{\omega_\rho\left(\frac{1}{1-\varepsilon}\right)}{\omega_\rho\left(\frac{1}{\varepsilon}\right)}\rho(h-f)$$

Thus,

$$\begin{aligned}\liminf_{k'\to\infty}\rho(T^{n_{k'}}g-h) &\le (1+\mu)br(f)-\frac{1}{\omega_\rho\left(\frac{1}{\varepsilon}\right)}\rho(T^{n_{k_0}}f-f)+\frac{\omega_\rho\left(\frac{1}{1-\varepsilon}\right)}{\omega_\rho\left(\frac{1}{\varepsilon}\right)}\rho(h-f)\\ &\le (1+\mu)br(f)-\frac{\gamma}{\omega_\rho\left(\frac{1}{\varepsilon}\right)}r(f)+\frac{\delta}{\omega_\rho\left(\frac{1}{\varepsilon}\right)}r(f)\\ &\le \alpha r(f).\end{aligned}$$

Therefore, in both cases we obtain $r(h)\le\alpha r(f)$.

Moreover

$$\begin{aligned}
\rho(h-f) &= \rho\left(2\left(\frac{h-f}{2}\right)\right)\\
&\le \omega(2)\rho\left(\frac{h-f}{2}\right)\\
&\le \omega(2)\big(\liminf_{k'\to\infty}\rho(h-T^{n_{k'}}g)+\liminf_{k'\to\infty}\rho(T^{n_{k'}}g-f)\big)\\
&\le \omega(2)\big(\alpha r(f)+(1+\mu)r(f)\big)\\
&= Ar(f).
\end{aligned}$$

By induction, we construct a sequence in the following way: we choose $h_0 = f$. If $h_0, h_1, ........, h_{n-1}$ are defined, we consider $h_n$ as the corresponding element for $h_{n-1}$ in the above construction. Thus,

$$r(h_n) \le \alpha r(h_{n-1}) \le ....... \le \alpha^n r(h_0)$$

and

$$\rho(h_{n+1}-h_n) \le Ar(h_n)$$

Hence, there exists an integer $N$ and some $\beta < 1$ such that for $n > N$ we have

$$\rho(h_{n+1}-h_n) \le K\alpha^n \le \beta^n,$$

which implies

$$\frac{1}{\beta^n} \le \frac{1}{\rho(h_{n+1}-h_n)}$$

and

$$\omega^{-1}\left(\frac{1}{\beta^n}\right) \le \omega^{-1}\left(\frac{1}{\rho(h_{n+1}-h_n)}\right).$$

Property(4) in lemma 4.1 implies

$$\left(\omega^{-1}\left(\frac{1}{\beta}\right)\right)^n \le \omega^{-1}\left(\frac{1}{\rho(h_{n+1}-h_n)}\right),$$

and from Lemma 4.3 we obtain

$$||h_{n+1}-h_n||_\rho \le \frac{1}{\omega^{-1}\left(\frac{1}{\rho(h_{n+1}-h_n)}\right)} \le \frac{1}{\left(\omega^{-1}(\frac{1}{\beta})\right)^n}.$$

Hence $\{h_n\}$ is a Cauchy sequence in $(L_\rho, ||.||_\rho)$ and there exists $h \in L_\rho$ such that $||h_n - h||_\rho \to 0$,because $(L_\rho, ||.||_\rho)$ is complete. Since under $\Delta_2$-type condition norm-convergence and modular-convergence are identical, $\{h_n\}$ is modular convergent to $h$. So,there exists a subsequence $\{g_n\}_n$ of $\{h_n\}_n$ such that $g_n \to h$ $(\rho - a.e)$ and $h \in C$ because $C$ is $\rho - a.e$ sequentially closed. We will prove that $r(h) = 0$.

Let $\varepsilon > 0$. We choose $n$ large enough such that $r(h_n) < \frac{\varepsilon}{2}$ and $\rho(h_n - h) < \frac{\varepsilon}{2}$. There exists $g \in C$ such that $\liminf\limits_{k\to\infty} \rho(T^{n_k}g - h_n) \leq \dfrac{\varepsilon}{2}$. Hence

$$\liminf_{k\to\infty} \rho\left(\frac{T^{n_k}g - h}{2}\right) \leq \liminf_{k\to\infty} \rho(T^{n_k}g - h_n) + \rho(h_n - h) \leq \varepsilon.$$

Thus

$$\liminf_{k\to\infty} \rho(T^{n_k}g - h) \leq \omega(2)\varepsilon$$

which implies

$$r(h) \leq \omega(2)\varepsilon.$$

Since $\varepsilon$ is arbitrary, we obtain

$$r(h) = 0.$$

Finally, we claim that

$$T(h) = h.$$

Indeed, choose an arbitrary $\varepsilon > 0$. There exists $g \in C$ such that $\liminf\limits_{k\to\infty} \rho(h - T^{n_k}g) < \varepsilon$. Therefore,

$$\begin{aligned}
\rho\left(\frac{1}{3}(h - T(h)\right) &\leq \liminf_{k\to\infty} \rho(h - T^{n_k}g) + \liminf_{k\to\infty} \rho(T^{n_k}g - T^{n_k+1}g) \\
&\qquad + |T| \liminf_{k\to\infty} \rho(T^{n_k}g - h) \\
&\leq (|T| + 1)\varepsilon.
\end{aligned}$$

Again, the arbitrarness of $\varepsilon$ implies $T(h) = h$.

**Remark 5.1.** In the following example we will prove that 2 is, in general, the best constant in Theorem 5.1.

**Example 5.1.** Assume that $L_\rho = l^1$ where $\rho(x) = ||x||$ for $x \in l^1$.
Then $\rho$-a.e convergence and weak star convergence are identical on bounded subsets of $l^1$.
Consider the set $B = \{x \in l^1; x_1 = 0, x_i \geq 0 \text{ if } i \geq 2 \text{ and } ||x|| \leq 1\}$
and

$$S^+ = \{x \in B : ||x|| = 1\}.$$

Since

$$B = B(0,1) \bigcap H_1 \bigcap \left( \bigcap_{n=2}^{\infty} H_n \right)$$

where $H_1 = \{x \in l^1 : x_1 = 0\}$ and $H_n = \{x \in l^1 : x_n \geq 0\}$ for $n \geq 2$, we know that $B$ is weakly star compact, i.e. $\rho$-a.e sequentially compact.
Define $S : B \to S^+$ by $S(x) = (1 - ||x||)e_1 + x$. Then, $S$ is well defined and $S$ is 2-Lipschitzian. Denote $R$ the right-shift operator in $l_1$, $R(x_1, x_2, .....) = (0, x_1, x_2, ....)$. Then $\dfrac{I+R}{2}$ is defined from $S^+$ to $S^+$. Furthermore, $T = \left(\dfrac{I+R}{2}\right) S$ defined from $B$ into $S^+ \subset B$ is 2-Lipschitzian.
Since $S$ is the identity on $S^+$, we have $T^n = \left(\dfrac{I+R}{2}\right)^n S$ which is 2-Lipschitzian.
Hence $T$ is 2-Uniformly Lipschitzian. By Isikhawa Theorem [12], $\dfrac{I+R}{2}$ is asymptotically regular and so $T$ is .
Finally, $T$ is fixed point free. Indeed, $T(x) = x$ implies $||x|| = 1$ and so $S(x) = x$. This implies $\dfrac{I+R}{2}x = x$, that is, $R(x) = x$, but $x = 0$ is the unique fixed point of $R$ in $l_1$ .

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