A spanning tree heuristic for partitioning a graph into centered connected components

V. Gratta, I. Lari, J. Puerto, F. Ricca, A. Scozzari

Seville, December 15\textsuperscript{th} 2014
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   - Application: political districting

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p-Centered Partition Problem [Apollonio et al. 2008]

**p-centered connected partition**

Given a graph $G=(V,E)$ and a subset $S$ of vertices $V$ called ”centers”, a p-centered partition is a partition into $p$ connected components where each component contains exactly one center.

**p-centered partition problem**

In the general p-centered partition problem we want to find a p-centered partition of the graph optimizing a cost-based objective function.

Application: clustering, image processing, territorial districting, etc. . .
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**Application: political districting** [Ricca, Scozzari, Simeone, 2013]

**Problem**

Design a district map of the given territory, represented as a contiguity graph ([Simeone, 1978]), and subdividing it into a fixed number of districts in which the election is performed.
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Notation

\[ T = (V, E) \] tree. \( |V| = n \)
\[ S \subseteq V \] centers. \( |S| = p < n \)
\[ U = V \setminus S \] units
\[ c : U \times S \rightarrow \mathbb{R} \] function that associates a cost \( c_{is} \geq 0 \) to each pair \((i, s), i \in U, s \in S\)

Problem

Find a p-centered partition of \( T \) that minimizes the maximum assignment cost of a unit \( i \in U \) to a center \( s \in S \).
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Mathematical programming formulation on T

Variables

\[ y_{is} = \begin{cases} 
1 & \text{if unit } i \text{ is assigned to center } s \\
0 & \text{otherwise} 
\end{cases} \quad i \in U, s \in S \]
Mathematical programming formulation on $T$

**Constraints**

$j(i, s)$: vertex $j$ that is adjacent to $i$ in the unique path from $i$ to $s$ in $T$

$$y_{is} \leq y_{j(i,s)s} \quad \forall i \in U, s \in S, (i, s) \notin E$$
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Mathematical programming formulation on $T$

Constraints

$$\sum_{s \in S} y_{is} = 1 \quad \forall i \in U$$

Each unit $i$ must be assigned to exactly one center $s$. 

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Objective function

$$\min \max \max_{s \in S} \max_{i \in U} c_{is} y_{is}$$

Minimize of the worst-case assigning cost.
Mathematical programming formulation on $T$

\[
\min_{s \in S} \max_{i \in U} \sum_{s \in S} y_{is} \quad \text{s.t.} \quad \sum_{s \in S} y_{is} = 1 \quad \forall i \in U
\]

\[
y_{is} \leq y_{j(i,s)} \quad \forall i \in U, s \in S, (i, s) \notin E
\]

\[
y_{is} \in \{0, 1\} \quad \forall i \in U, s \in S
\]
Mathematical programming formulation (Feasibility Problem)

Given a fixed value $\alpha$, find, if exists, a p-centered partition of $T$ such that
\[
\max_{s \in S} \max_{i \in U} c_{is}y_{is} \leq \alpha
\]

\[
y_{is} \leq y_{j(i,s)} \quad \forall i \in U, s \in S, (i, s) \notin E
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\sum_{s \in S} y_{is} = 1 \quad \forall i \in U
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y_{is} \in \{0, 1\} \quad \forall i \in U, s \in S
\]

\[
y_{is} = 0 \quad \text{if } c_{is} > \alpha, i \in U, s \in S
\]
Mathematical programming formulation (Relaxed Feasibility Problem)

Given a fixed value $\alpha$, find, if exists, a $p$-centered partition of $T$ such that
$$\max_{s \in S} \max_{i \in U} c_{is} y_{is} \leq \alpha$$

$$y_{is} \leq y_{j(i,s)} \quad \forall i \in U, s \in S, (i, s) \notin E$$

$$\sum_{s \in S} y_{is} = 1 \quad \forall i \in U$$

$$y_{is} \geq 0 \quad \forall i \in U, s \in S$$

$$y_{is} = 0 \quad \text{if} \ c_{is} > \alpha, i \in U, s \in S$$

This is a Linear Programming problem and his feasible polytope has integer vertices ([Lari, Puerto, Ricca, Scozzari, 2014]).
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Algorithm [Lari, Puerto, Ricca, Scozzari, 2014]

1. Sort the $c_{is}$ values, $i \in U$, $s \in S$, in non-decreasing order
2. Apply a binary search to generate all the possible different values
   $\alpha = \min_{s \in S} \max_{i \in U} \max_{s \in S} c_{is} y_{is}$ of problem (1).
3. For each $\alpha$ solve the feasibility problem (3).
Algorithm [Lari, Puerto, Ricca, Scozzari, 2014]

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<th>2</th>
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Problem on a graph

We know that the problem is NP-hard on general graphs, but we have a polynomial time algorithms for trees ([Lari, Puerto, Ricca, Scozzari, 2014]).

Idea

Exploit the exact algorithm on trees to solve heuristically the problem on general graphs, basing on the correspondence that exists between the optimal partition of a graph and that of one of its spanning trees. ([Maravalle e Simeone, 1995]).
**Basic idea of the heuristic algorithm**

1. Generate a spanning tree of $G$, $T = (V, E_T)$.
2. Apply to $T$ the polynomial time algorithm for trees.
3. Modify locally $T$ to obtain a new spanning tree of $G$, $T' \neq T$.
4. Update $T$ with $T'$ and go to 2.
Example

$$C = \begin{pmatrix}
6 & 13 \\
6 & 6 \\
9 & 10 \\
11 & 12 \\
5 & 4 \\
1 & 4 \\
14 & 8 \\
10 & 1 \\
5 & 4 \\
6 & 8 \\
4 & 7 \\
9 & 14 \\
8 & 8 \\
6 & 6 \\
2 & 1 \\
13 & 7 \\
7 & 6 \\
11 & 13
\end{pmatrix}$$
Example
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I. Lari, J. Puerto, F. Ricca, A. Scozzari, 2014
Partitioning a graph into connected components with fixed centers and optimizing different criteria.
submitted to the scientific journal Networks.

F. Ricca, A. Scozzari, B. Simeone, 2011
Political districting: from classical models to recent approaches.

Polynomial Algorithms for Partitioning a Tree into Single-Center Subtrees to Minimize Flat Service Costs.
References

M. Maravalle, B. Simeone, 1995
A spanning tree heuristic for regional clustering.

B. Simeone, 1978
Optimal graph partitioning.
Atti giornate di lavoro AIRO, Urbino. pp 57–73, 1978;
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