Exploiting symmetries in multifacility location

Seminario Nuevos Desafíos de la Matemática Combinatoria
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Outline

1. The Theory of Moments
2. Continuos 1-F OM Location
3. Continuos MF OM Location
Moments, SOS and SDP

Let's \( K = \{ x \in \mathbb{R}^d : g_j(x) \geq 0, \forall j \}; \tilde{f}, g_j \in \mathbb{R}[X] \).

\[
\begin{align*}
\min_{x \in K} \tilde{f}(x) \\
\end{align*}
\]
Moments, SOS and SDP

Let’s $K = \{ x \in \mathbb{R}^d : g_j(x) \geq 0, \forall j \}; \tilde{f}, g_j \in \mathbb{R}[X]$.

$$\min_{x \in K} \tilde{f}(x) \quad (\text{Lasserre, 2001}) \equiv \begin{cases} \min_{\mu \in \mathcal{M}(K)} & \int \tilde{f} \, d\mu \\ \int d\mu = 1 \\ \mu \geq 0 \end{cases},$$

(1)

$\mathcal{M}(K)$ is the vector space of finite, signed Borel measures supported on $K$. 

\[ \mathbb{R}[X] = \mathbb{R}[1, X_1, \ldots, X_d]. \]

\[ B = [1, X_1, \ldots, X_d, X_1^2, X_1X_2, \ldots, X_1X_d, X_2^2, \ldots] \]

\[ = [X^\alpha]_{\alpha \in \mathbb{N}^d} \]
\[ \mathbb{R}[X] = \mathbb{R}[1, X_1, \ldots, X_d]. \]

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\[ = [X^\alpha]_{\alpha \in \mathbb{N}^d} \]

Then, for any sequence \( \{y_\alpha\}_\alpha \) indexed in the same order that \( \mathcal{B} \), and for any polynomial \( \tilde{f} = \sum_{\alpha \in \mathbb{N}^d} \tilde{f}_\alpha X^\alpha \) We introduce the functional \( L_y : \mathbb{R}[X] \rightarrow \mathbb{R} : \)

\[ \tilde{f} \mapsto \sum_{\alpha \in \mathbb{N}^d} \tilde{f}_\alpha y_\alpha \]
Moments, SOS and SDP

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\[ \tilde{f} \mapsto \sum_{\alpha \in \mathbb{N}^d} \tilde{f}_\alpha y_\alpha \]

**Example:** For \( \mathbb{R}[x_1, x_2] \) and \( r = 2 \), \( \mathcal{B} = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, \ldots\} \), so for \( f = x_1^2 - x_2^3 + 2x_1x_2 \in \mathbb{R}[x_1, x_2] \): \( L_y(f) = y_{2,0} - y_{0,3} + 2y_{1,1} \)
The moment matrix of order $r$, $M_r(y)$, is defined as $M_r(y)_{ij} = L_y(b_i \cdot b_j)$, where $b_i$ are the elements in $\mathcal{B}$ such that $\text{deg}(b_i), \text{deg}(b_j) \leq r$. 
The moment matrix of order \( r \), \( M_r(y) \), is defined as \( M_r(y)_{ij} = L_y(b_i \cdot b_j) \), where \( b_i \) are the elements in \( B \) such that \( \text{deg}(b_i), \text{deg}(b_j) \leq r \).

\[
M_2(y) = \begin{pmatrix}
1 & X_1 & X_2 & X_1^2 & X_1X_2 & X_2^2 \\
1 & y_0,0 & y_1,0 & y_0,1 & y_1,1 & y_0,2 \\
x_1 & y_1,0 & y_2,0 & y_1,1 & y_2,1 & y_1,2 \\
x_2 & y_0,1 & y_1,1 & y_0,2 & y_1,2 & y_0,3 \\
x_1^2 & y_2,0 & y_3,0 & y_2,1 & y_3,1 & y_2,2 \\
x_1x_2 & y_1,1 & y_2,1 & y_1,2 & y_2,2 & y_1,3 \\
x_2^2 & y_0,2 & y_1,2 & y_0,3 & y_1,3 & y_0,4
\end{pmatrix}
\]
For any $g \in \mathbb{R}[X]$, the \textbf{localizing matrix} of $g$ of order $r$, $M_r(g \cdot y)$, is defined as $M_r(g \cdot y)_{ij} = L_y(g \cdot b_i \cdot b_j)$, where $b_i$ are the elements in $B$ such that $\text{deg}(g) + \text{deg}(b_i) + \text{deg}(b_j) \leq 2r$. 
For any $g \in \mathbb{R}[X]$, the **localizing matrix** of $g$ of order $r$, $M_r(g \cdot y)$, is defined as

$$M_r(g \cdot y)_{ij} = L_y(g \cdot b_i \cdot b_j),$$

where $b_i$ are the elements in $B$ such that $\deg(g) + \deg(b_i) + \deg(b_j) \leq 2r$.

For $g(x_1, x_2) = 1 - x_1 x_2$ and $r = 1$:

$$M_1(g \cdot y) = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_1,0 - y_{1,1} & y_{1,1} - y_{1,2} \\ x_2 & x_2,0 - y_{2,1} & y_{2,1} - y_{2,2} \end{pmatrix}$$
If $K$ satisfies Arquimedean Property (also called Putinar’s Property):

$$\exists u \in \mathbb{R}[x] : \{x : u(x) \geq 0\} \text{ is compact and}$$

$$u = \sigma_0 + \sum_{j=1}^{\ell} \sigma_j g_j, \text{ being } \sigma_j \text{ s.o.s. polynomials} \tag{2}$$
Moments, SOS and SDP

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Then,

$$\inf \begin{cases} \langle \tilde{f}, \mu \rangle \\ s.t. \quad \langle 1, \mu \rangle = 1, \\ \mu \geq 0, \\ \mu \in \mathcal{M}(K) \end{cases} \equiv \inf \begin{cases} L_y(\tilde{f}) \\ s.t. \quad \{y_\alpha\} \text{ has} \\ \text{a representing} \\ \text{measure} \end{cases}$$

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Moments, SOS and SDP

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(2)

Then,

$$\inf \begin{bmatrix} \langle \tilde{f}, \mu \rangle \\ \text{s.t. } \langle 1, \mu \rangle = 1, \mu \geq 0, \mu \in \mathcal{M}(K) \end{bmatrix} \equiv \inf \begin{bmatrix} L_y(\tilde{f}) \\ \text{s.t. } \{y_\alpha\} \text{ has a representing measure} \end{bmatrix} \equiv \inf \begin{bmatrix} L_y(\tilde{f}) \\ \text{s.t. } M_r(y) \succeq 0, M_r(g_j y) \succeq 0, \forall j, \forall r \geq \nu \end{bmatrix}$$

$$\nu = \max\{\left\lfloor \frac{\deg \tilde{f}}{2} \right\rfloor, \left\lfloor \frac{\deg g_j}{2} \right\rfloor\}$$
Moments, SOS and SDP

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$$\exists u \in \mathbb{R}[x] : \{ x : u(x) \geq 0 \} \text{ is compact and}$$

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Then,

$$\inf \langle \tilde{f}, \mu \rangle \quad \text{s.t.} \quad \langle 1, \mu \rangle = 1, \mu \geq 0, \mu \in \mathcal{M}(K) \equiv \inf L_y(\tilde{f}) \quad \text{s.t.} \quad \{ y_\alpha \} \text{ has a representing measure} \equiv \inf L_y(\tilde{f}) \quad \text{s.t.} \quad M_r(y) \succeq 0, \forall j, \forall r \geq \nu$$

$$\nu = \max \{ \left\lfloor \frac{\deg \tilde{f}}{2} \right\rfloor, \left\lfloor \frac{\deg g_j}{2} \right\rfloor \}$$

Solving for a fixed $r_0$ it gives a relaxation of the original polynomial optimization problem by a "SDP"
Continuos 1-F OM Location
Continuos 1-F OM Location
Generalized location problems with rational objective

We are given \( n \) point \( \{a_1, \ldots, a_n\} \subset \mathbb{R}^d \) endowed with an \( \ell_\alpha \)-norm, \( \alpha = r/s \in \mathbb{Q}, \) m.c.d.\((r, s) = 1.\)

- \( f_j := \frac{p_j}{q_j} : \mathbb{R}^d \rightarrow \mathbb{R} \) are polynomials or piecewise polynomials for \( j = 1, \ldots, m.\)
Generalized location problems with rational objective

We are given $n$ point $\{a_1, \ldots, a_n\} \subset \mathbb{R}^d$ endowed with an $\ell_\alpha$-norm, $\alpha = r/s \in \mathbb{Q}$, m.c.d.($r$, $s$) = 1.

- $f_j := \frac{p_j}{q_j} : \mathbb{R}^d \to \mathbb{R}$ are polynomials or piecewise polynomials for $j = 1, \ldots, m$.

- $K = \{(x, u) \in \mathbb{R}^d \times \mathbb{R}_+^n : g_j(x, u) \geq 0, j = 1, \ldots, m, q_j > 0 \}

$u_i = \begin{cases} \\
\left\{ \begin{array}{ll}
\|x - a_i\|_\alpha, &\text{or} \\
\min \|x - a_i\|_\alpha, &\text{if } i = 1, \ldots, n \\
\end{array} \right. 
\end{cases}$

In most cases, we add the redundant constraint $\|x\|_2^2 \leq M$. $K$ is a closed, compact semi-algebraic set.
We shall define the dependence of \( f_j \) to the decision variable \( x \in \mathbb{R}^d \) via \( u = (u_1, \ldots, u_n) \), where \( u_i : \mathbb{R}^d \mapsto \mathbb{R}, u_i(x) := \|x - a_i\|_\alpha, i = 1, \ldots, n \). Therefore, the \( j \)-th component of the ordered median objective function of our problems reads as:

\[
\tilde{f}_j(x) : \mathbb{R}^d \mapsto \mathbb{R} \\
x \mapsto \tilde{f}_j(x) := f_j(\|x - a_1\|_\alpha, \ldots, \|x - a_n\|_\alpha)
\]

Consider the following problem:

\[
(\text{LOCOMRF}) \quad \rho_\lambda := \min_x \left\{ \sum_{j=1}^m \lambda_j(x) \tilde{f}_j(x) : x \in K \right\},
\]

- \( K \subseteq \mathbb{R}^d \) satisfies Putinar’s property.
• \( f(u_1, \ldots, u_n) = \sum_{i=1}^{n} u_i \) if
\[
u_i = \begin{cases} \|x - a_i\|_\alpha, & \text{Weber problem} \\ \min_{x \in X, |X| = p} \|x - a_i\|_\alpha, & \text{Multifacility Weber problem} \end{cases}
\]
• \( f(u_1, \ldots, u_n) = \sum_{i=1}^{n} u_i^q \) the centroid problem,
• \( f(u_1, \ldots, u_n) = \max_{1 \leq i \leq n} u_i \equiv \min_{u_i \leq z} z \), center problem,
• \( f(u_1, \ldots, u_n) = \sum_{i=1}^{n} \lambda_i u(i) \equiv \min_{\sum_{i=1}^{n} \lambda_1 u_{\sigma(i)} \leq z, \forall \sigma} z \), ordered median problem,
• $f(u_1, \ldots, u_n) = \sum_{i<j}^{n} |u_i - u_j|$, absolute deviation or envy problem

• $f(u_1, \ldots, u_n) = \sum_{i=1}^{n} (u_i - \bar{u})^2$, variance problem ...

• $f(u_1, \ldots, u_n) = \sum_{j=1}^{n} \frac{w_j}{u_j^2}$, obnoxious facility location

• $f(u_1, \ldots, u_n) = \sum_{j=1}^{n} \frac{b_j}{1 + h_j |u_j|^\lambda}$, Huff competitive location

• Gradual covering, acceleration-deceleration distance, inventory gradual covering ...
Equivalent problem

\[ \bar{\rho}_\lambda = \min_{x, w, u, v} \sum_{j=1}^{m} \lambda_j(x) \sum_{i=1}^{m} f_i(u) w_{ij} \]  (4)

s.t. \[ \sum_{j=1}^{m} w_{ij} = 1, \text{ for } i = 1, \ldots, m \]

\[ \sum_{i=1}^{m} w_{ij} = 1, \text{ for } j = 1, \ldots, m \]

\[ \sum_{i=1}^{m} w_{ij} f_i(u) \geq \sum_{i=1}^{m} w_{ij+1} f_i(u), j = 1, \ldots, m \]  (5)

\[ \sum_{i=1}^{m} \sum_{j=1}^{m} w_{ij}^2 - w_{ij} = 0, \]

\[ v_{ij}^s \geq x_j - a_{ij}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, d \]  (6)

\[ v_{ij}^s \geq a_{ij} - x_j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, d \]  (7)

\[ z_i^r = \sum_{j=1}^{d} v_{ij}^r, \quad i = 1, \ldots, n, \]  (8)

\[ u_i = z_i^s, \quad i = 1, \ldots, n, \]

\[ \sum_{i,j=1}^{m} w_{ij}^2 \leq m, \quad w_{ij} \in \mathbb{R}, \quad x \in K. \]
For $r \geq \max\{r_0, \nu_0\}$ where $r_0 := \max_{k=1,\ldots,\ell} \xi_k$ and $\nu_0 := \max\{\max_{j=0,\ldots,m} \nu_j, \max_{j=1,\ldots,m} \nu'_j\} = \max_{j=0,\ldots,m} \nu_j$, we introduce the following hierarchy of semidefinite programs:

$$\begin{align*}
\min_y & \quad L_y(p_\lambda) \\
\text{s.t.} & \quad M_r(y; l(0)) \succeq 0, \\
& \quad M_r - \xi_k (g_k y; l(0)) \succeq 0, \quad k = 1, \ldots, \ell, \\
& \quad M_r (y; l(0) \cup l(j) \cup l(j+1)) \succeq 0, \quad j = 1, \ldots, m, \\
& \quad M_{r-\nu_j} (h_j y; l(0) \cup l(j) \cup l(j+1)) \succeq 0, \quad j = 1, \ldots, m-1, \\
& \quad M_{r-1} (h'_j y; l(0) \cup l(j) \cup l(j+1)) \succeq 0, \quad j = 1, \ldots, m, \\
& \quad L_y (\sum_{i=1}^m w_{ij} - 1) = 0, \quad j = 1, \ldots, m, \\
& \quad L_y (\sum_{j=1}^m w_{ij} - 1) = 0, \quad i = 1, \ldots, m, \\
& \quad L_y (w_{ij}^2 - w_{ij}) = 0, \quad i, j = 1, \ldots, m, \\
& \quad L_y (q_\lambda) = 1,
\end{align*}$$

with optimal value denoted $\min Q_r$. 

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**Symmetries in MF OMP**  
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Some Results

Theorem (B., Puerto, ElHaj-BenAli, 2012)

Let \((x)\) be a feasible solution of \((\text{LOCOMF})\) then there exists a solution \((x, u, v, w)\) for \((\text{MFOMP1}_\lambda)\) such that their objective values are equal. Conversely, if \((x, u, v, w)\) is a feasible solution for \((\text{MFOMP1}_\lambda)\) then there exists a solution \((x)\) for \((\text{LOCOMF})\) having the same objective value. In particular \(\varrho_\lambda = \overline{\varrho}_\lambda\). Moreover, if \(K \subset \mathbb{R}^d\) satisfies Putinar’s property then \(\overline{K} \subset \mathbb{R}^{d+m^2+n(d+2)}\) also satisfies Putinar’s property.
Some Results

Theorem (B., Puerto, ElHaj-BenAli, 2012)

1. Let \((x)\) be a feasible solution of (LOCOMF) then there exists a solution \((x, u, v, w)\) for (MFOMP1_\lambda) such that their objective values are equal. Conversely, if \((x, u, v, w)\) is a feasible solution for (MFOMP1_\lambda) then there exists a solution \((x)\) for (LOCOMF) having the same objective value. In particular \(\rho_\lambda = \overline{\rho}_\lambda\). Moreover, if \(K \subset \mathbb{R}^d\) satisfies Putinar’s property then \(\overline{K} \subset \mathbb{R}^{d+m^2+n(d+2)}\) also satisfies Putinar’s property.

2. Let \(\overline{K} \subset \mathbb{R}^{d+m^2+n(d+2)}\) (compact) be the feasible domain of Problem (MFOMP1_\lambda). Let \(Q_r\) be the semidefinite program (Q1_r) with \((g_k), (h_j) \subset \mathbb{R}[x, u, v, w]\) the polynomial functions defining the constraints of \(\overline{K}\). Then:
   (a) \(\inf Q_r \uparrow \rho\) as \(r \to \infty\).
   (b) Let \(y^r\) be an optimal solution of the SDP relaxation \(Q_r\) in (Q1_r). If

   \[
   \text{rank } M_r(y^r) = \text{rank } M_{r-r_0}(y^r) = t
   \] \hspace{1cm} (9)

   then \(\min Q_r = \rho\) and one may extract \(t\) points \((x^*(k), u^*(k), v^*(k), w^*(k))_{k=1}^t \subset \overline{K}\), all global minimizers of the MOMRF problem.

B. & ElHaj-BenAli & Puerto, CORS 2013
Convex OMP

\[
\min_{x \in \mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{\sigma(i)} \|x - a_{\sigma(i)}\|_\tau.
\] (10)

Theorem (B., Puerto, ElHaj-BenAli, 2013)

For any set of lambda weights satisfying \(\lambda_1 \geq \ldots \geq \lambda_n\) and \(\tau = rs\) such that \(r, s \in \mathbb{N} \setminus \{0\}\), \(r > s\) and \(\gcd(r, s) = 1\), Problem (10) can be represented as a semidefinite programming problem with \(n^2 + n(2d + 1)\) linear constraints and at most \(4nd \log r\) positive semidefinite constraints.
\[
\min_{x \in \mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{\sigma(i)} \| x - a_{\sigma(i)} \|_{\tau}.
\]

\textbf{Theorem (B., Puerto, ElHaj-BenAli, 2013)}

For any set of lambda weights satisfying \( \lambda_1 \geq \ldots \geq \lambda_n \) and \( \tau = \frac{r}{s} \) such that \( r, s \in \mathbb{N} \setminus \{0\} \), \( r > s \) and \( \gcd(r, s) = 1 \), Problem (10) can be represented as a semidefinite programming problem with \( n^2 + n(2d + 1) \) linear constraints and at most \( 4nd \log r \) positive semidefinite constraints.

B. & Puerto & ElHaj-BenAli, Preprint 2013
Constrained Case

Theorem
Consider the restricted problem:

\[
\min_{x \in \mathcal{K} \subset \mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{\sigma(i)} \| x - a_{\sigma(i)} \|_\tau.
\] (11)

Assume that the hypothesis of Theorem 2 holds. In addition, any of the following conditions holds:

1. \(g_i(x)\) are concave for \(i = 1, \ldots, \ell\) and \(-\sum_{i=1}^{\ell} \mu_i \nabla^2 g_i(x) \succeq 0\) for each dual pair \((x, \mu)\) of the problem of minimizing any linear functional \(c^t x\) on \(\mathcal{K}\) (Positive Definite Lagrange Hessian (PDLH)).

2. \(g_i(x)\) are sos-concave on \(\mathcal{K}\) for \(i = 1, \ldots, \ell\) or \(g_i(x)\) are concave on \(\mathcal{K}\) and strictly concave on the boundary of \(\mathcal{K}\) where they vanish, i.e. \(\partial \mathcal{K} \cap \partial \{x \in \mathbb{R}^d : g_i(x) = 0\}\), for all \(i = 1, \ldots, \ell\).

3. \(g_i(x)\) are strictly quasi-concave on \(\mathcal{K}\) for \(i = 1, \ldots, \ell\).

Then, there exists a constructive finite dimension embedding, which only depends on \(\tau\) and \(g_i, i = 1, \ldots, \ell\), such that (18) is a semidefinite problem.
Constrained Case

\[(Q_N) : \min \sum_{k=1}^{n} v_k + \sum_{i=1}^{n} w_i\]  
\[s.t. \quad v_i + w_k \geq \lambda_k z_i, \quad \forall i, k = 1, \ldots, n,\]  
\[y_{ij} - x_j + a_{ij} \geq 0, \quad \forall i = 1, \ldots, n, j = 1, \ldots, d.\]  
\[y_{ij} + x_j - a_{ij} \geq 0, \quad \forall i = 1, \ldots, n, j = 1, \ldots, d.\]  
\[y_{ij}^r \leq u_{ij}^s z_i^{r-s}, \quad \forall i = 1, \ldots, n, j = 1, \ldots, d,\]  
\[\omega_i^r \sum_{j=1}^{d} u_{ij} \leq z_i, \quad \forall i = 1, \ldots, n,\]  
\[M_N(\kappa) \geq 0,\]  
\[M_{N-\xi_k}(g_k, \kappa) \geq 0, \quad k = 1, \ldots, \ell,\]  
\[L_\kappa(x_j) = x_j, \quad j = 1, \ldots, d,\]  
\[L_\kappa(z_i) = z_i, \quad i = 1, \ldots, n,\]  
\[L_\kappa(v_i) = v_i, \quad i = 1, \ldots, n,\]  
\[L_\kappa(w_i) = w_i, \quad i = 1, \ldots, n,\]  
\[L_\kappa(u_{ij}) = u_{ij}, \quad i = 1, \ldots, n, j = 1, \ldots, d,\]  
\[L_\kappa(y_{ij}) = y_{ij}, \quad i = 1, \ldots, n, j = 1, \ldots, d,\]  
\[\kappa_0 = 1\]  
\[u_{ij} \geq 0, \quad \forall i = 1, \ldots, n, j = 1, \ldots, d.\]  

with optimal value denoted \(\min Q_N\).
Constrained Case

Theorem

Consider \( \rho_\lambda \) defined as the optimal value of the problem:

\[
\rho_\lambda = \min_{x \in K \subset \mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{\sigma(i)} \| x - a_{\sigma(i)} \|_\tau.
\]  

(20)

Then, with the notation above:

(a) \( \min Q_N \uparrow \rho_\lambda \) as \( N \to \infty \).

(b) Let \( \kappa^N \) be an optimal solution of Problem \((Q_N)\). If

\[
\text{rank} \, M_N(\kappa^N) = \text{rank} \, M_{N-N_0}(\kappa^r) = \vartheta
\]

then \( \min Q_N = \rho_\lambda \) and one may extract \( \vartheta \) points

\[
(x^*(i), z^*(i), v^*(i), w^*(i), u^*(i), y^*(i))_{i=1}^{\vartheta} \subset K,
\]

all global minimizers of Problem (27).
Theorem

Consider $\rho_\lambda$ defined as the optimal value of the problem:

$$
\rho_\lambda = \min_{x \in K \subset \mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{\sigma(i)} \| x - a_{\sigma(i)} \|_\tau.
$$

(20)

Then, with the notation above:

(a) $\min Q_N \uparrow \rho_\lambda$ as $N \to \infty$.

(b) Let $\kappa_N^N$ be an optimal solution of Problem $(Q_N)$. If

$$
\text{rank } M_N(\kappa_N^N) = \text{rank } M_{N-N_0}(\kappa^r) = \vartheta
$$

then $\min Q_N = \rho_\lambda$ and one may extract $\vartheta$ points

$$(x^*(i), z^*(i), v^*(i), w^*(i), u^*(i), y^*(i))_{i=1}^{\vartheta} \subset K,$$

all global minimizers of Problem (27).

B. & Puerto & ElHaj-BenAli, Preprint 2013
Symmetries in MF OMP

Priego de Córdoba 2013

V. Blanco
Continuos MF OM Location
We are given a set of demand points \( S = \{a_1, \ldots, a_n\} \) and two sets of scalars \( \Omega := \{\omega_1, \ldots, \omega_n\}, \omega_i \geq 0, \forall i \in \{1, \ldots, n\} \) and \( \Lambda := \{\lambda_1, \ldots, \lambda_n\} \) where \( \lambda_1 \geq \ldots \geq \lambda_n \geq 0 \). The elements \( \omega_i \) are weights corresponding to the importance given to the existing facilities \( a_i, i \in \{1, \ldots, n\} \).
\[
\rho_\lambda := \min_x \left\{ \sum_{i=1}^{n} \lambda_i \tilde{f}_i(x) : x = (x_1, \ldots, x_p), \ x_j \in K, \ \forall j = 1, \ldots, p \right\}, \quad (\text{LOCOMF})
\]

where:

- \( K \subseteq \mathbb{R}^d \) satisfies the Archimedean property. Without loss of generality we shall assume that we know \( M > 0 \) such that \( \sum_{j=1}^{p} \|x_j\|_2^2 \leq M \).
- \( \tau := \frac{r}{s} \geq 1, \ r, s \in \mathbb{N}, \ r \geq s \) and \( \gcd(r, s) = 1 \).
- \( \lambda_\ell \geq 0 \) for all \( \ell = 1, \ldots, n \).
\[
\overline{p}_\lambda = \min_{x,y,w,u,v} \sum_{\ell=1}^{n} \lambda_\ell \sum_{i=1}^{n} t_i w_{i\ell}
\]

\[
\text{s.t. } h_1^i := \sum_{\ell=1}^{n} w_{i\ell} - 1 = 0, \quad \text{for } i = 1, \ldots, n,
\]

\[
h_2^\ell := \sum_{i=1}^{n} w_{i\ell} - 1 = 0, \quad \text{for } \ell = 1, \ldots, n,
\]

\[
h_3^\ell := \sum_{i=1}^{n} w_{i\ell} t_i - \sum_{i=1}^{n} w_{i\ell+1} t_i \geq 0, \quad \ell = 1, \ldots, n - 1,
\]

\[
h_4^i := w_{i1}^2 - w_{i\ell} = 0, \quad \text{for } i, \ell = 1, \ldots, n,
\]

\[
h_5^\ell := 1 - \sum_{i=1}^{n} w_{i\ell}^2 \geq 0, \quad \ell = 1, \ldots, n
\]

\[
h_{ij}^{6k} := v_{ijk}^s - (x_{jk} - a_{ik})^r \geq 0, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p, \quad k = 1, \ldots, d
\]

\[
h_{ij}^{7k} := v_{ijk}^s - (a_{ik} - x_{jk})^r \geq 0, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p, \quad k = 1, \ldots, d
\]

\[
h_{ij}^{8}\left(\sum_{k=1}^{d} v_{ijk}\right)^s - u_{ij}^r \geq 0, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p
\]

\[
h_{ij}^{9} := u_{ij} - t_i \geq 0, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p
\]

\[
h_{ij}^{10} := t_i - z_{ij} u_{ij} \geq 0, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p
\]

\[
h_{ij}^{11} := \sum_{j=1}^{p} z_{ij} - 1 = 0, \quad i = 1, \ldots, n
\]

\[
h_{ij}^{12} := z_{ij}^2 - z_{ij} \geq 0, \quad \text{for } i = 1, \ldots, n, \quad j = 1, \ldots, p
\]
The p-median Euclidean case

\[ \min_{x=(x_1,\ldots,x_p)\in\mathbb{R}^{pd}} \sum_{i=1}^{m} w_i \min_{j=1,\ldots,p} \|x_j-a_i\|_2 \quad \equiv \quad \begin{cases} 
\min \left( \sum_{i=1}^{n} w_i t_i \right) \\
\text{s.t.:} \\
t_i^2 \geq \sum_{j=1}^{p} \sum_{k=1}^{d} z_{ij}(a_{ik} - x_{jk})^2, \quad i = 1, \ldots, n \\
\sum_{j=1}^{p} z_{ij} = 1, \quad i = 1, \ldots, n \\
\sum_{j=1}^{p} (z_{ij} - z_{ij}^2) \leq 0, \quad i = 1, \ldots, n \\
\sum_{i=1}^{n} t_i^2 + \sum_{i=1}^{p} \sum_{j=1}^{p} z_{ij}^2 + \sum_{j=1}^{p} \sum_{k=1}^{d} x_{jk}^2 \leq M \\
t_i \geq 0, \quad z_{ij} \in [0,1], \quad x_j \in K, \quad \forall i = 1, \ldots, n, \quad j = 1, \ldots, p. \end{cases} \]
Theorem
Let $x$ be a feasible solution of $\text{LOCOMF}$ then there exists a solution $(x, z, u, v, w, t)$ for $\text{MFOMP}_{1, \lambda}$ such that their objective values are equal. Conversely, if $(x, z, u, v, w, t)$ is a feasible solution for $\text{MFOMP}_{1, \lambda}$ then there exists a solution $(x)$ for $\text{LOCOMF}$ having the same objective value. In particular $\rho_\lambda = \overline{\rho}_\lambda$. Moreover, if $K \subset \mathbb{R}^d$ satisfies Archimidean’s property then $\overline{K} \subset \mathbb{R}^{pd+np+n+nd+n^2+n}$ also satisfies Archimidean’s property.
The Moment approach

Let \( h_0(x, z, u, v, w, t) := p_\lambda(x, w, t) \), and denote \( \xi_j := \lceil (\text{deg } g_j)/2 \rceil \) and \( \nu_j := \lceil (\text{deg } h_j)/2 \rceil \), where \( \{g_1, \ldots, g_{n_K}\} \), and \( \{h_0, h_1, \ldots, h_{nc1}\} \) are, respectively, the polynomial constraints that define \( K \) and \( \overline{K} \setminus K \) in \( \text{MFOMP1}_\lambda \). For \( r \geq r_0 := \max \{ \max_{k=1, \ldots, n_K} \xi_k, \max_{j=0, \ldots, nc} \nu_j \} \), introduce the hierarchy of semidefinite programs:

\[
\begin{align*}
\min_y & \quad L_y(p_\lambda) \\
\text{s.t.} & \quad M_r(y) \succeq 0, \\
& \quad M_{r-\xi_k}(g_k, y) \succeq 0, \quad k = 1, \ldots, n_K, \\
& \quad M_{r-\nu_j}(h_j, y) \succeq 0, \quad j = 1, \ldots, nc1, \\
& \quad y_0 = 1,
\end{align*}
\]  

(Q1\( r \))

with optimal value denoted \( \inf Q1_r \) (and \( \min Q1_r \) if the infimum is attained).
The Moment approach

**Theorem**

Let $\overline{K} \subset \mathbb{R}^{pd+np+npd+n^2+n}$ (compact) be the feasible domain of Problem $MFOMP1_\lambda$. Let $\inf Q_1 r$ be the optimal value of the semidefinite program $Q_1 r$. Then, with the notation above:

(a) $\inf Q_1 r \uparrow \rho_\lambda$ as $r \to \infty$.

(b) Let $y^r$ be an optimal solution of the SDP relaxation $Q_1 r$. If

$$\text{rank } M_r(y^r) = \text{rank } M_{r-n_0}(y^r) = \varphi$$

then $\min Q_1 r = \rho_\lambda$ and one may extract $\varphi$ points

$$(x_1^*(k), \ldots, x_p^*(k), z^*(k), u^*(k), v^*(k), w^*(k), t^*(k))_{k=1}^{\varphi} \subset \overline{K}, \text{ all global minimizers of the }$$

$MFOMP1_\lambda \text{ problem.}$
SOC Programming Formulation

\[
\hat{\rho}_\lambda = \min \sum_{\ell=1}^{n} \lambda_\ell \theta_\ell
\]

s.t. (28), (29), (38),

\[
t_i \leq \theta_\ell + UB_i (1 - w_i \ell), \quad i = 1, \ldots, n, \quad \ell = 1, \ldots, n,
\]

\[
\theta_\ell \geq \theta_{\ell+1}, \quad \ell = 1, \ldots, n-1,
\]

\[
u_{ijk} - x_{jk} + a_{ik} \geq 0, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p, \quad k = 1, \ldots, d,
\]

\[
u_{ijk} + x_{jk} - a_{ik} \geq 0, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p, \quad k = 1, \ldots, d,
\]

\[
\nu_{ijk}^r \leq d_{ijk}^s u_{ij}^{r-s}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p, \quad k = 1, \ldots, d,
\]

\[
\sum_{k=1}^{d} d_{ijk} \leq u_{ij}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p,
\]

\[
w_{i \ell} \in \{0, 1\}, \quad \theta_\ell \in \mathbb{R}^+, \quad \forall i, \ell = 1, \ldots, n,
\]

\[
z_{ij} \in \{0, 1\}, \quad \forall i = 1, \ldots, n, \quad j = 1, \ldots, p,
\]

\[
t_i \in \mathbb{R}^+, \quad v_{ijk}, d_{ijk} \in \mathbb{R}^+, \quad u_{ij} \in \mathbb{R}^+, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p, \quad k = 1, \ldots, d,
\]

\[
x_j \in \mathcal{K}, \quad j = 1, \ldots, p.
\]

\[
(MFOMP2_\lambda)
\]

Theorem

Let \( x \) be a feasible solution of \( LOCOMF \) then there exists a solution \( (x, z, u, v, w, t, \theta, \varsigma, d) \) for \( MFOMP2_\lambda \) such that their objective values are equal. Conversely, if \( (x, z, u, v, w, t, \theta, \varsigma, d) \) is a feasible solution for \( MFOMP2_\lambda \) then there exists a solution \( (x) \) for \( LOCOMF \) having the same objective value. \( \rho_\lambda = \hat{\rho}_\lambda. \)
The p-median Euclidean case

\[
\min_{x \in \mathbb{R}^{pd}} \sum_{i=1}^{m} w_i \min_{j=1, \ldots, p} \| x_j - a_i \|_2 \quad \equiv \quad \begin{cases} 
\min \sum_{i=1}^{n} w_i t_i \\
s.t.: \quad u_{ij}^2 \geq \sum_{k=1}^{d} (a_{ik} - x_{jk})^2, & i = 1, \ldots, n; \ j = 1, \ldots, p, \\
& i = 1, \ldots, n; \ j = 1, \ldots, p, \\
& \sum_{j=1}^{p} z_{ij} \geq 1, & i = 1, \ldots, n \\
& z_{ij} \in \{0, 1\}, & i = 1, \ldots, n; \ j = 1, \ldots, p, \\
& t_i \geq 0, \ u_{ij} \geq 0, & i = 1, \ldots, n; \ j = 1, \ldots, p, \\
& x = (x_1, \ldots, x_p) \in \mathbb{R}^{pd}. & 
\end{cases}
\]
Let $y = (y_\alpha)$ be a real sequence indexed in the monomial basis $(x^\beta z^\eta u^\gamma v^\delta w^\zeta t^\alpha \theta^\varsigma d^\psi)$ of $\mathbb{R}[x, z, u, v, w, t, \theta, d]$ (with 
$$\alpha = (\beta, \eta, \gamma, \delta, \zeta, \alpha, \varsigma, \psi) \in \mathbb{N}^{pd} \times \mathbb{N}^{np} \times \mathbb{N}^{np} \times \mathbb{N}^{npd} \times \mathbb{N}^n \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}^{npd}).$$

Let $h_0(\theta) := \sum_{\ell=1}^m \lambda_\ell \theta_\ell$, and denote $\xi_j := \lceil (\deg g_j)/2 \rceil$ and $\nu_j := \lceil (\deg h_j)/2 \rceil$, where 
$$\{g_1, \ldots, g_{n_K}\}, \text{ and } \{h_1, \ldots, h_{nc2}\}$$
are, respectively, the polynomial constraints that define $K$ and $\hat{K} \setminus K$ in MFOMP2$\lambda$. For $r \geq r_0 := \max\{ \max_{k=1,\ldots,n_K} \xi_k, \max_{j=0,\ldots,nc2} \nu_j \}$, introduce the hierarchy of semidefinite programs:

$$\min_y \quad L_y(p_\lambda)$$

s.t. 
$$M_r(y) \succeq 0,$$
$$M_{r-\xi_k}(g_k, y) \succeq 0, \quad k = 1, \ldots, n_K,$$
$$M_{r-\nu_j}(h_j, y) \succeq 0, \quad j = 1, \ldots, nc2,$$
$$y_0 = 1,$$

with optimal value denoted $\inf Q2_r$ (and $\min Q2_r$ if the infimum is attained).
The SDP-relaxation Approach

Theorem

Let \( \hat{K} \subset \mathbb{R}^{d+np(d+2)+n^2+2n+npd} \) (compact) be the feasible domain of Problem MFOMP2\(\lambda\). Let \( \inf Q_2_r \) be the optimal value of the semidefinite program \( Q_2_r \). Then, with the notation above:

(a) \( \inf Q_2_r \uparrow \rho_\lambda \) as \( r \to \infty \).

(b) Let \( y^r \) be an optimal solution of the SDP relaxation \( Q_2_r \). If

\[
\text{rank } M_r(y^r) = \text{rank } M_{r-r_0}(y^r) = \varphi
\]

then \( \min Q_2_r = \rho_\lambda \) and one may extract \( \varphi \) points

\( (x_1^*(k), \ldots, x_p^*(k), z^*(k), u^*(k), v^*(k), w^*(k), t^*(k), \theta^*(k), d^*(k))_{\varphi=1}^{\varphi} \subset \hat{K} \), all global minimizers of the MFOMP2\(\lambda\) problem.
The SDP-relaxation Approach

Theorem
Let \( \hat{K} \subset \mathbb{R}^{d+np(d+2)+n^2+2n+npd} \) (compact) be the feasible domain of Problem MFOMP2\(\lambda\). Let \( \inf Q_{2r} \) be the optimal value of the semidefinite program \( Q_{2r} \). Then, with the notation above:

(a) \( \inf Q_{2r} \uparrow \rho_\lambda \) as \( r \to \infty \).
(b) Let \( y^r \) be an optimal solution of the SDP relaxation \( Q_{2r} \). If

\[
\text{rank } M_r(y^r) = \text{rank } M_{r-r_0}(y^r) = \varphi
\]

then \( \min Q_{2r} = \rho_\lambda \) and one may extract \( \varphi \) points

\((x_1^*(k), \ldots, x_p^*(k), z^*(k), u^*(k), v^*(k), w^*(k), t^*(k), \theta^*(k), d^*(k))_{k=1}^\varphi \subset \hat{K}, \) all global minimizers of the MFOMP2\(\lambda\) problem.

Bottleneck: \( N = n^2 + 2np + pd + n + npd \) variables \( \Rightarrow \) SDP Matrix Size:

\[
\binom{N + r}{r}
\]
Reduction (I): Sparsity

Assuming that:

1. There is $M > 0$ such that $\|x\|_2^2 < M$ for all $x \in \mathbf{K}$.
2. The index sets $I = \{1, \ldots, d\}$ and $J = \{1, \ldots, m\}$ are partitioned into sets $\{I_k\}_{k=1}^{\pi}$ and $\{J_k\}_{k=1}^{\pi}$ respectively, satisfying:
   1. $\{J_\ell\}$ are disjoint sets.
   2. For every $j \in J_k$, the constraint $g_j(x) \geq 0$ is only concerned with the variables $X(I_k) = \{x_i : i \in I_k\}$.
   3. The objective function $f$ can be written as $f = \sum_{k=1}^{\pi} f_k$ where $f_k \in \mathbb{R}[X(I_k)]$ for $k = 1, \ldots, \pi$.
   4. For every $k = 1, \ldots, \pi - 1$ $I_{k+1} \cap \bigcup_{j=1}^{k} I_j \subseteq I_s$ for some $s \leq k$. 
Reduction (I): Sparsity

Then, we consider the following semidefinite program:

\[
Q_r^{sp} : \inf_y L_y(f) \quad \begin{align*}
M_r(y; l_k) & \succeq 0, \quad k = 1, \ldots, \pi, \\
M_{r-\lceil\deg g_j/2\rceil}(g_j y; l_k) & \succeq 0, \quad j \in J_k, \quad k = 1, \ldots, \pi, \quad 1 \leq j \leq m \\
y_0 & = 1
\end{align*}
\] (48)

Theorem (Lasserre, 2006)

With the notation above, \( \lim_{r \to \infty} \inf Q_r^{sp} = \min \{ f(x) : x \in K \} \). Furthermore, if \( y^r \) is a feasible solution of \( Q_r^{sp} \) with \( L_{y^r}(f) \leq \inf Q_r^{sp} + \frac{1}{r} \) and \( \hat{y}^r = \{ y^r_\alpha : \sum_{i=1}^r \alpha_i = 1 \} \), then \( \lim_{r \to \infty} \hat{y}^r = x^* \), if \( x^* \in K \) is the unique global minimizer of the polynomial optimization problem.
Let \( \tilde{I}(0, 0) = I^x \cup I^w \cup I^t \) and \( \tilde{I}(j, \ell) = I^x(j) \cup I^z(j) \cup I^u(j) \cup I^v(j) \cup I^w(\ell) \cup I^t \) for all \( \ell = 1, \ldots, n - 1, j = 1, \ldots, p \).
Let $\tilde{I}(0, 0) = I_x \cup I_w \cup I_t$ and $\tilde{I}(j, \ell) = I_x(j) \cup I_z(j) \cup I_u(j) \cup I_v(j) \cup I_w(\ell) \cup I_t$ for all $\ell = 1, \ldots, n - 1$, $j = 1, \ldots, p$.

Observe that

$$\tilde{I}(j + 1, \ell + 1) \cap \bigcup_{j' \leq j, \ell' \leq \ell} \tilde{I}(j, \ell) \subseteq \tilde{I}(0, 0), \quad \forall j \geq 0, \ \ell \geq 0.$$  \hfill (49)
Reduction (I): Sparsity

For $r \geq \max\{r_0, \nu_0\}$ where $r_0 := \max_{k=1,\ldots,\ell} \xi_k$ and $\nu_0 := \max_{j=0,\ldots,12} \nu^j_{\ell}$:

$$\inf_y L_y \left( \sum_{\ell=1}^{n} \sum_{i=1}^{n} \lambda_i(x) t_i w_{i\ell} \right)$$

subject to

- $M_r(y; \tilde{I}(0, 0)) \succeq 0$,
- $M_{r-\xi_k}(g_k y; \tilde{I}(0, 0)) \succeq 0$, $k = 1, \ldots, n_k$,
- $M_r(y; \tilde{I}(j, \ell)) \succeq 0$, $j = 1, \ldots, p$, $\ell = 1, \ldots, n - 1$,
- $M_{r-\nu^3_{\ell}}(h^3_{\ell} y; \tilde{I}(j, \ell)) \succeq 0$, $j = 1, \ldots, p$, $\ell = 1, \ldots, n - 1$,
- $M_{r-\nu^5_{\ell}}(h^5_{\ell} y; \tilde{I}(j, \ell)) \succeq 0$, $j = 1, \ldots, p$, $\ell = 1, \ldots, n - 1$,
- $M_{r-\nu^6_{ijk}}(h^6_{ijk} y; \tilde{I}(j, \ell)) \succeq 0$, $i = 1, \ldots, n$, $j = 1, \ldots, p$, $k = 1, \ldots, d$, $\ell = 1, \ldots, n - 1$,
- $M_{r-\nu^7_{ijk}}(h^7_{ijk} y; \tilde{I}(j, \ell)) \succeq 0$, $i = 1, \ldots, n$, $j = 1, \ldots, p$, $k = 1, \ldots, d$, $\ell = 1, \ldots, n - 1$,
- $M_{r-\nu^8_{ijk}}(h^8_{ijk} y; \tilde{I}(j, \ell)) \succeq 0$, $i = 1, \ldots, n$, $j = 1, \ldots, p$, $k = 1, \ldots, d$, $\ell = 1, \ldots, n - 1$,
- $M_{r-\nu^9_{ij}}(h^9_{ij} y; \tilde{I}(j, \ell)) \succeq 0$, $i = 1, \ldots, n$, $j = 1, \ldots, p$, $\ell = 1, \ldots, n - 1$,
- $M_{r-\nu^{10}_{ij}}(h^{10}_{ij} y; \tilde{I}(j, \ell)) \succeq 0$, $i = 1, \ldots, n$, $j = 1, \ldots, p$,
- $L_y \left( \sum_{i=1}^{n} w_{i\ell} - 1 \right) = 0$, $\ell = 1, \ldots, n$,
- $L_y \left( \sum_{\ell=1}^{n} w_{i\ell} - 1 \right) = 0$, $i = 1, \ldots, n$,
- $L_y \left( w_{i\ell} - w_{i\ell} \right) = 0$, $i, \ell = 1, \ldots, n$,
- $L_y \left( \sum_{j=1}^{p} z_{ij} - 1 \right) = 0$, $i, \ell = 1, \ldots, n$,
- $L_y \left( z_{ij}^2 - z_{ij} \right) = 0$, $i = 1, \ldots, n$, $j = 1, \ldots, p$,

with optimal value denoted $\inf Q_{1_r}^{sp}$. 

(Q$_{1_r}^{sp}$)

Symmetries in MF OMP

Priego de Córdoba 2013

V. Blanco
**Theorem**

Let \( K \subset \mathbb{R}^{pd+n^2+np+npd+n^2+n} \) be the feasible domain of \( \text{MFOMP}_{1\lambda} \). Then, with the notation above:

(a) \( \inf Q_1^{sp} \uparrow \rho_\lambda \) as \( r \to \infty \).

(b) Let \( y^r \), be an optimal solution of the SDP relaxation \( Q_1^{sp} \). If

\[
\begin{align*}
\text{rank } M_r(y^r; l^x) &= \text{rank } M_{r-r_0}(y^r; l^x) \\
\text{rank } M_r(y^r; \tilde{I}(j, \ell)) &= \text{rank } M_{r-v_0}(y^r; \tilde{I}(j, \ell)) \quad \ell = 1, \ldots, n, j = 1, \ldots, p
\end{align*}
\]

and if

\[
\text{rank}(M_r(y^r; l^x \cup (I^z(j) \cup I^u(j) \cup I^v(\ell) \cup I^t) \cap (I^z(j') \cup I^u(j') \cup I^v(\ell') \cup I^t))) = 1
\]

for all \( (j, \ell) \neq (j', k') \) then \( \inf Q_1^{sp} = \rho_\lambda \).

Moreover, let \( \Delta_{j,\ell} := \{(x^*(j, \ell), z^*(j, \ell), u^*(j, \ell), v^*(j, \ell), w^*(j, \ell)), t^*(j, \ell)\} \) be the set of solutions obtained by the application of the condition (57). Then, every \((x^*, z^*, u^*, v^*, w^*, t^*)\) such that \((x^*_{jk}, z^*_{ij}, u^*_{ij}, v^*_{ij}, w^*_{ij}, t^*_i)_{(i,j,k) \in \tilde{I}(j',k') = (x^*(j', k'), z^*(j', k'), u^*(j', k'), v^*(j', k'), w^*(j', k'), t^*(j', k'))}\) for some \( \Delta_{j', k'} \) is an optimal solution of Problem \( \text{MOMRF}_{\lambda} \).
We will apply the symmetry results when permuting the \( j \)-indices in the set of variables \( \Upsilon = \{ x, z, u, v \} \).
Reduction (II): Symmetry

We will apply the symmetry results when permuting the $j$-indices in the set of variables $\mathcal{Y} = \{x, z, u, v\}$.

We consider the following action $\varphi$ over $\mathbb{R}^p$:

$$\varphi : S_p \times \mathbb{R}^p \to \mathbb{R}^p$$

defined as $\varphi(\sigma, (y_1, \ldots, y_p)) = (y_{\sigma(1)}, \ldots, y_{\sigma(p)})$ for any $\sigma \in S_p$ and $y \in \mathbb{R}^p$. 
We will apply the symmetry results when permuting the \( j \)-indices in the set of variables \( \Upsilon = \{ x, z, u, v \} \).

We consider the following action \( \varphi \) over \( \mathbb{R}^p \):

\[
\varphi : S_p \times \mathbb{R}^p \rightarrow \mathbb{R}^p
\]

defined as \( \varphi(\sigma, (y_1, \ldots, y_p)) = (y_{\sigma(1)}, \ldots, y_{\sigma(p)}) \) for any \( \sigma \in S_p \) and \( y \in \mathbb{R}^p \).

\[
\varphi_\Upsilon : S_p \times \mathbb{R}^{Np+M} \rightarrow \mathbb{R}^{Np+M}
\]

defined such that \( \varphi_\Upsilon \) maps \( (\sigma, (x, z, u, v, w, t)) \) into

\( (\varphi(\sigma, x(I_x(1))), \ldots, \varphi(\sigma, v(I^v(i, k))), w(I^w), t(I^t)) \), i.e., permuting the indices associated with facilities in the decision variables (the \( j \)-index).
Reduction (II): Symmetry

We will apply the symmetry results when permuting the $j$-indices in the set of variables $\Upsilon = \{x, z, u, v\}$.

We consider the following action $\varphi$ over $\mathbb{R}^p$:

$$\varphi : S_p \times \mathbb{R}^p \rightarrow \mathbb{R}^p$$

defined as $\varphi(\sigma, (y_1, \ldots, y_p)) = (y_{\sigma(1)}, \ldots, y_{\sigma(p)})$ for any $\sigma \in S_p$ and $y \in \mathbb{R}^p$.

$$\varphi_\Upsilon : S_p \times \mathbb{R}^{Np+M} \rightarrow \mathbb{R}^{Np+M}$$

defined such that $\varphi_\Upsilon$ maps $(\sigma, (x, z, u, v, w, t))$ into

$$(\varphi(\sigma, x(I^x(1))), \ldots, \varphi(\sigma, v(I^v(i, k))), w(I^w), t(I^t)),$$

i.e., permuting the indices associated with facilities in the decision variables (the $j$-index).

$$\varphi_\Upsilon = \varphi \oplus \cdots \oplus \varphi \oplus \mathbf{1}_M.$$

Symmetries in MF OMP

Priego de Córdoba 2013

V. Blanco
By Maschke’s Theorem (see [?, Thm 1.5.3]), every $G$-module $V$ is a direct sum of irreducible $G$-submodules of $V$, i.e.,

$$V \cong \bigoplus_{i=1}^{s} V_i$$

with irreducible $G$-submodules $V_i$. (51)

Each irreducible $G$-submodule might occur several times in the direct sum.
\begin{equation}
\inf_y L^{\text{sym}}(p, y) \\
Q_r^{\text{sym}}: \quad M_r^{\text{sym}}(y) \succeq 0, \\
M_r^{\text{sym}}(g_j y) \succeq 0, \quad 1 \leq j \leq m
\end{equation}

with optimal value denoted by \( \inf Q_r^{\text{sym}} \).

**Theorem ([?])**

*Assume that the Archimedean Property holds and let \((Q_r^{\text{sym}})_{r \geq r_0}\) be the hierarchy of SDP-relaxations defined in (59). Then \((\inf Q_r^{\text{sym}})_{r \geq r_0}\) is a monotone non-decreasing sequence that converges to \( \rho^* \).*
Lemma

Let $B_k(Y)$ be a symmetry-adapted basis of $\mathbb{R}[Y_1, \ldots, Y_p]$ of degree at most $k$ and $B^\text{st}(X)$ the standard monomial basis of $\mathbb{R}[w(l^w), t(l^t)]$ with degree at most $k$. Then, the elements of a symmetry-adapted basis of $\mathbb{R}[x, z, u, v, w, t]$ are of the form:

$$b = b^{x_1} \cdots b^{v_{n,d}} \cdot b'$$

where $b^{x_k} \in B_k(x(l^x(k)))$, $b^{z_i} \in B_i(z(l^z(i)))$, $b^{u_{i}} \in B_k(u(l^u(i)))$, $b^{v_{i,k}} \in B_k(v(l^v(i,k)))$, for $i = 1, \ldots, n$, $k = 1, \ldots, d$, and $b' \in B^\text{st}(X)$ and such that $\deg(b) \leq k$. 
Lemma

Let $T$ be a generalized Young tableau with shape $\lambda \vdash p$ and content $\mu^\beta$. The generalized Specht polynomials $S_{(t_\lambda, T)}$ generate an $S_p$-submodule of $\mathbb{R}\{Y^\beta\}$ which is isomorphic to the Specht module $S^\lambda$.

With the above results, we get the following result which is proven in [?].

Theorem

Let $\beta \in \mathbb{N}_0^p$ with $\sum_{i=1}^{p} \beta_i = r$ and shape $\mu^\beta$. Then:

$$
\mathbb{R}\{Y^\beta\} = \bigoplus_{\lambda \supseteq \mu^\beta} \bigoplus_{T \in T_{\lambda, \mu}} \mathbb{R}\{S_{t_\lambda, T}\}
$$

where $t_\lambda$ denotes the unique $\lambda$-tableau with increasing rows and columns and $T_{\lambda, \mu}$ the set of semistandard generalized Young tableaux of shape $\lambda$ and content $\mu$. 
Example

For \( n = 3 \) demand points, in the plane \( (d = 2) \), \( p = 2 \) facilities to be located and relaxation order \( k = 2 \):
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Construct a symmetry-adapted basis for $S_2$ over a set of two variables.
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First, the components in the symmetry-adapted basis are indexed by the partitions $\lambda \vdash (2)$, thus $\lambda \in \{(2), (1, 1)\}$. The $\beta$ to take into account are $\beta \in \{(0, 0), (1, 0), (2, 0), (1, 1)\}$ with shapes $\mu$ equal to $(2)$, $(1, 1)$, $(1, 1)$ and $(2)$, respectively. Thus, the semistandard generalized Young tableaux for each of these shapes and contents are:

- $\mu = (2)$: \[
\begin{array}{cc}
1 & 1 \\
1 & 2
\end{array}
\]
- $\mu = (1, 1)$: No semistandard generalized Young tableaux exists in this case.
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Hence, there is only one irreducible component in this case (for $\mu = (2)$), so the symmetry-adapted basis in $\mathbb{R}[Y_1, Y_2]$ is

$$\{1, Y_1 + Y_2, Y_1^2 + Y_2^2, Y_1 Y_2\}$$

We observe that the standard monomial basis for this set of two variables has 6 monomials while this basis has only four elements.