Weighted norm inequalities for singular integral operators

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Abstract

For a Calderón–Zygmund singular integral operator $T$, we show that the following weighted inequality holds

$$\int_{\mathbb{R}^n} |Tf(y)|^p w(y) dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M^{[p]+1} w(y) dy,$$

where $M^k$ is the Hardy–Littlewood maximal operator $M$ iterated $k$ times, and $[p]$ is the integer part of $p$. Moreover, the result is sharp since it does not hold for $M^p$.

We also give the following endpoint result:

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| M^2 w(y) dy.$$
1 Introduction and statements of the results

A classical result due to C. Fefferman and E. Stein [4] states that the Hardy–Littlewood maximal operator $M$ satisfies the following inequality for arbitrary $1 < p < \infty$, and weight $w$

$$\int_{\mathbb{R}^n} |Mf(y)|^p w(y) dy \leq C \int_{\mathbb{R}^n} |f(y)|^p Mw(y) dy,$$

where $C$ is independent of $f$. A weight $w$ in $\mathbb{R}^n$ will always be a nonnegative locally integrable function.

The study of weighted inequalities like the above, for other operators has played a central rôle in modern of Harmonic Analysis since they appear in duality arguments. We refer the reader to [5] Chapters 5 and 6 for a very nice exposition.

Although we could work with any Calderón–Zygmund operator (cf. §3), we shall only consider singular integral operators of convolution type defined by:

$$Tf(x) = p.v. \int_{\mathbb{R}^n} k(x - y)f(y)\, dy,$$

where the kernel $k$ is $C^1$ away from the origin, has mean value on the unit sphere centered at the origin and satisfies for $y \neq 0$

$$|k(y)| \leq \frac{C}{|y|^n} \quad \text{and} \quad |\nabla k(y)| \leq \frac{C}{|y|^{n+1}}.$$

It is well known that the analogous version of inequality (1) fails for the Hilbert transform for all $p$. In [3] A. Córdoba and C. Fefferman have shown that there is a similar inequality for any $T$, but with $Mw$ replaced by the pointwise larger operator $M_r w = M(w^r)^{1/r}$, $r > 1$, that is, for $1 < p < \infty$

$$\int_{\mathbb{R}^n} |Tf(y)|^p w(y) dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M_r w(y) dy,$$

with $C$ independent of $f$.

The purpose of this paper is to prove weighted norm inequalities of the form (2), where $M_r w$, $r > 1$, will be replaced by appropriate smaller maximal–type operators $w \rightarrow Nw$ satisfying

$$Mw(x) \leq Nw(x) \leq C M_r w(x),$$
for each $x \in \mathbb{R}^n$. We shall also be concern with corresponding endpoints results such as weak type $(1, 1)$ and $H^1-L^1$ estimates.

Before stating our main results, we shall make the following observation. Let $M^k$ be the Hardy–Littlewood maximal operator $M$ iterated $k$ times, where $k = 1, 2, \cdots$. We claim that for $k = 2, \cdots$, and $r > 1$, there exists a positive constant $C$ independent of $w$ such that

$$Mw(x) \leq M^k w(x) \leq CM_r w(x), \quad (4)$$

for each $x \in \mathbb{R}^n$. The left inequality follows from the Lebesgue differentiation theorem; for the other, we let $B$ be the best constant in Coifman’s estimate $M(M_r w) \leq B M_r w$, where $B$ is independent of $w$. Then, it follows easily that $M^k w \leq B^{k-1} M_r w$, $k = 1, 2, \cdots$.

In view of this observation, it is natural to consider whether or not (2) holds for some $M^k$, with $k = 2, 3, \cdots$. In a very interesting paper [8], M. Wilson has recently obtained the following partial answer to this question: Let $1 < p < 2$, then

$$\int_{\mathbb{R}^n} |Tf(y)|^p w(y)dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M^2 w(y)dy, \quad (5)$$

Moreover, he shows that this estimate does not hold for $p \geq 2$, and also that when $p = 2$, $M^2 w$ can be replaced by $M^3 w$. However, his method does not yield corresponding estimates for $p > 2$ (cf. §3 of that paper), and $M^2 w$ must be replaced by a much more complicated expression.

M. Wilson’s approach to this problem is based on certain (difficult) estimates for square functions that he obtained in the same paper, together with a couple of related estimates for the area function, obtained essentially by S. Chanillo and R. Wheeden in [1].

In this paper we give a complete answer to Wilson’s problem by means of a different method. Our main result is the following.

**Theorem 1.1:** Let $1 < p < \infty$, and let $T$ be a singular integral operator. Then, there exists a constant $C$ such that for each weight $w$

$$\int_{\mathbb{R}^n} |Tf(y)|^p w(y)dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M^{[p]+1} w(y)dy, \quad (6)$$

where $[p]$ is the integer part of $p$. Furthermore, the result is sharp since it does not hold for $M^{[p]}$.

The corresponding weak–type $(1, 1)$ version of this result is the following.
Theorem 1.2: Let $T$ be a singular integral operator. Then, there exists a constant $C$ such that for each weight $w$ and for all $\lambda > 0$

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| M^2 w(y) dy.$$  

(7)

Remark 1.3: Let $1 < p < \infty$, a natural question is whether (7) can be extended to the case $(p, p)$, that is whether

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(y)|^p M^p w(y) dy,$$
holds for some constant $C$ and for all $\lambda > 0$. At the end of section 2 we give an example showing that this inequality is false when $p$ is not an integer; however, we do not know what happens when $p$ is an integer.

Although we do not know whether (7) holds for $Mw$ (cf. remark 1.7) we can give the following estimate. For a measure $\mu$ we shall denote by $H^1(\mu)$ the subspace of $L^1(\mu)$ of functions $f$ which can be written as $f = \sum_j \lambda_j a_j$, where $a_j$ are $\mu$–atoms and $\lambda_j$ are complex numbers with $\sum_j |\lambda_j| < \infty$. A function $a$ is a $\mu$–atom if there is a cube $Q$ for which $supp(a) \subset Q$, so that

$$|a(x)| \leq \frac{1}{\mu(Q)},$$
and

$$\int_Q a(y) dy = 0.$$

Theorem 1.4: Let $T$ be a singular integral operator. Then, there exists a constant $C$ such that for each weight $w$

$$\int_{\mathbb{R}^n} |Tf(y)| w(y) dy \leq C \|f\|_{H^1(Mw)}.$$  

(8)

Theorem 1.1 is in fact a consequence of a more precise estimate than (6). The idea is to replace the operator $M^{|p|+1}$ by an optimal class of maximal operators. We explain now what “optimal” means.

We want to define a scale of maximal–type operators $w \rightarrow M_A w$ such that

$$Mw(x) \leq M Aw(x) \leq M_r w(x)$$
for each $x \in \mathbb{R}^n$, where $r > 1$. $A$ stands for a Young function; i.e. $A : [0, \infty) \rightarrow [0, \infty)$ is continuous, convex and increasing satisfying $A(0) = 0$. To define $M_A$ we introduce for each cube $Q$ the $A$–average of a function $f$ over $Q$ by means of the following Luxemburg norm

$$\|f\|_{A,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|} \int_Q A\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1\}.$$

We define the maximal operator $M_A$ by

$$M_Af(x) = \sup_{x \in Q} \|f\|_{A,Q},$$

where $f$ is a locally integrable functions, and where the supremum is taken over all the cubes containing $x$. When $A(t) = t^r$ we get $M_A = M_r$, but more interesting examples are provided by Young functions like $A(t) = t \log(1 + t), \epsilon > 0$.

The optimal class of Young functions $A$ is characterized by the following theorem.

**Theorem 1.5:** Let $1 < p < \infty$, and let $T$ be a singular integral operator. Suppose that $A$ is a Young function satisfying the condition

$$\int_c^\infty \left(\frac{t}{A(t)}\right)^{p'} dt < \infty,$$

for some $c > 0$. Then, there exists a constant $C$ such that for each weight $w$

$$\int_{\mathbb{R}^n} |Tf(y)|^p w(y)dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M_Aw(y)dy. \tag{10}$$

Furthermore, condition (9) is also necessary for (10) to hold for all the Riesz transforms: $T = R_1, R_2, \cdots, R_n$.

We recall that the $j$–th Riesz transform $R_j, j = 1, 2, \cdots, n$, is the singular integral operator defined by

$$R_jf(x) = p.v. \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) dy.$$

The proof of this theorem is given in §2, and it is based on the following inequality of E.M. Stein [7]

$$\int_Q w(y) \log^k(1 + w(y)) dy \leq C \int_Q Mw(y) \log^{k-1}(1 + Mw(y)) dy, \tag{11}$$
with $k = 1, 2, 3, \ldots$.

As for the strong case, there is an estimate sharper than (7).

**Theorem 1.6:** Let $T$ be a singular integral operator. For arbitrary $\epsilon > 0$, consider the Young function

$$A_\epsilon(t) = t \log^\epsilon(1 + t).$$

(12)

Then, there exists a constant $C$ such that for each weight $w$ and for all $\lambda > 0$

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| M_A w(y) dy.$$

(13)

**Remark 1.7:** For $1 < p < \infty$ let us denote by $B_p$ the collection of all Young functions $A$ satisfying condition (9):

$$\int_c^\infty \left( \frac{t}{A(t)} \right)^{p'-1} \frac{dt}{t} < \infty,$$

for some $c > 0$. Observe that $B_p \subset B_q$, $1 < p < q < \infty$. Then it follows easily from the proof of last theorem that we may replace $A_\epsilon$ by any Young function belonging to the smallest class $\cap_{p>1} B_p$. We could consider for instance

$$A_\epsilon(t) = t \log(1 + t)|\log \log(1 + t)|^\epsilon.$$

(14)

If we let $\epsilon = 0$ in (12) $M_{A_0} = M$ is the Hardy–Littlewood maximal operator. Since $A_0$ does not belong to $\cap_{p>1} B_p$ we think that the estimate:

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| M w(y) dy,$$

(15)

for some constant $C$, and for all $\lambda > 0$, does not hold.

2 Proof of the Theorems

**Proof of Theorem 1.5:**

We prove first that condition (9) is sufficient for (10) to hold for any singular integral operator $T$.

We may assume that $M_A w$ is finite almost everywhere, and we let $T^*$ be the adjoint operator of $T$. $T^*$ is also a singular integral operator with kernel $k^*(x) = k(-x)$. Then, by duality (10) is equivalent to
Proof of the Theorems

\[ \int_{\mathbb{R}^n} |T^* f(y)|^{p'} M_A w(y)^{1-p'} dy \leq C \int_{\mathbb{R}^n} |f(y)|^{p'} w(y)^{1-p'} dy. \]  

(16)

We shall be using some well known facts about the $A_p$ theory of weights for which we remit the reader to [5] Chapter 4.

To prove (16) we shall use the following fundamental estimate due to Coifman ([2]):

Let $T$ be any singular integral operator; then for each $0 < p < \infty$, and each $u \in A_\infty$, there exists $C = C_{u,p} > 0$ such that for each $f \in C_0^\infty(\mathbb{R}^n)$

\[ \int_{\mathbb{R}^n} |T f(y)|^p u(y) dy \leq C \int_{\mathbb{R}^n} M f(y)^p u(y) dy. \]  

(17)

Therefore, to apply this estimate to $T^*$ we need to show that $(M_A w)^{1-p'}$ satisfies the $A_\infty$ condition.

To check this, we claim first that $(M w)^{\delta}$ satisfies the $A_1$ condition for $0 < \delta < 1$. However, this is a straightforward generalization of the well known fact that $(M w)^{\delta} \in A_1, 0 < \delta < 1$, also due to Coifman (cf. [5] p. 158), and we shall omit its proof.

Now, since $w^{1-r} \in A_r$, for any $w \in A_1$ and $r > 1$, we have that

\[ (M_A w)^{1-p'} = \left[ (M_A w)^{\frac{p'}{r-1}} \right]^{1-r} \in \cap_{r>p'} A_r \subset A_\infty. \]

After these observations, we have reduced the problem to showing that

\[ \int_{\mathbb{R}^n} M f(y)^p M_A w(y)^{1-p'} dy \leq C \int_{\mathbb{R}^n} |f(y)|^{p'} w(y)^{1-p'} dy. \]  

(18)

But this is a particular instance of the following characterization which can be found in [6] Theorem 4.4.

**Theorem 2.1:** Let $1 < p < \infty$. Let $A$ be a Young function, and denote $B = \overline{A(t^{p'})}$. Then the following are equivalent.

i) \[ \int_{c}^{\infty} \left( \frac{t}{A(t)} \right)^{p-1} \frac{dt}{t} < \infty; \]  

(19)

ii) there is a constant $c$ such that

\[ \int_{\mathbb{R}^n} M_B f(y)^p dy \leq c \int_{\mathbb{R}^n} f(y)^p dy \]  

(20)
for all nonnegative, locally integrable functions $f$;
iii) there is a constant $c$ such that
\[
\int_{\mathbb{R}^n} \mathcal{M}_B f(y)^p \ u(y) dy \leq c \int_{\mathbb{R}^n} f(y)^p \ M u(y) dy
\]  
(21)
for all nonnegative, locally integrable functions $f$ and $u$;
iv) there is a constant $c$ such that
\[
\int_{\mathbb{R}^n} \mathcal{M} f(y)^p \ \frac{u(y)}{[\mathcal{M}_A(w)(y)]^{p-1}} dy \leq c \int_{\mathbb{R}^n} f(y)^p \ \frac{M u(y)}{w(y)^{p-1}} dy,
\]  
(22)
for all nonnegative, locally integrable functions $f$, $w$ and $u$.

Observe that (18) follows from (22) by taking $u = 1$, and by replacing $p$ by $p'$.

Now we shall prove that condition (9) is also necessary for (10) to hold for all the Riesz transforms. That is, suppose that the Young function $A$ is fixed, and that the inequality
\[
\int_{\mathbb{R}^n} |Tf(x)|^p \ w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \ M_A w(x) dx,
\]  
(23)
is verified for each Riesz transform $T = R_j$, $j = 1, 2, \cdots, n$.

Fix one of these $j$. As above, by duality (23) is equivalent to
\[
\int_{\mathbb{R}^n} |R_j f(x)|^p' \ M_A w(x)^{1-p'} dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p'} \ w(x)^{1-p'} dx,
\]  
(24)
We shall adapt an argument from [5] p. 561. We define the cone
\[
E_j = \{x \in \mathbb{R}^n : \max\{|x_1|, |x_2|, \cdots, |x_n|\} = x_j\},
\]
so that $\mathbb{R}^n = \bigcup_{j=1}^n (E_j \cup (-E_j))$. Let $B$ be the unit ball, and consider the function $f = w = \chi_{B\cap(-E_j)}$. Then, (24) implies
\[
\infty > C \int_{\mathbb{R}^n} |f(x)|^{p'} \ w(x)^{1-p'} dx = C |B \cap (-E_j)| \geq
\]
\[
\geq \int_{E_j \cap \{|x| > 2\}} |R_j f(x)|^{p'} \ M_A f(x)^{1-p'} dx.
\]
Observe that for $|x| > 2$, $M_A f(x) \approx A^{-1}(|x|^n)^{-1}$. Also, for every $x \in E_j$

$$R_j f(x) = C \int_{B \cap (-E_j)} \frac{x_j - y_j}{|x - y|^{n+1}} dy \geq C \int_{B \cap (-E_j)} \frac{1}{|x - y|^n} dy \geq \frac{C}{|x|^n}.$$  

Therefore

$$\int_{E_j \cap \{|x| > 2\}} \frac{1}{|x|^{np'}} A^{-1}(|x|^n)^{p'-1} dx \leq C |B \cap (-E_j)|.$$  

A corresponding estimate can be proved for $E_j$, and for each $j = 1, 2, \ldots, n$, by using in each case the corresponding Riesz transform. Since the family of cones $\{-E_j\}_{j=1,2,\ldots,n}$ is disjoint, we finally have that

$$\int_{|x| > 2} \frac{1}{|x|^{np'}} A^{-1}(|x|^n)^{p'-1} dx \approx \int_{\infty}^{\infty} \frac{1}{t^{p'}} A^{-1}(t)^{p'-1} dt \approx \int_{\infty}^{\infty} \left( \frac{t}{A(t)} \right)^{p'-1} dt < \infty,$$

since $tA'(t) \approx A(t)$. This concludes the proof of the theorem.

\[ \square \]

**Proof of Theorem 1.6:**

We shall assume that $M_{A, w}$ is finite almost everywhere, since otherwise there is nothing to be proved.

For $f \in C_0^\infty(\mathbb{R}^n)$ we consider the standard Calderón–Zygmund decomposition of $f$ at level $\lambda$ (cf. [5] p. 414).

Let $\{Q_j\}$ be the Calderón–Zygmund nonoverlapping dyadic cubes satisfying

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} |f(x)| \, dx \leq 2^n \lambda. \quad (25)$$

If we let $\Omega = \bigcup_j Q_j$, we also have that $|f(x)| \leq \lambda$ a.e. $x \in \mathbb{R}^n \setminus \Omega$.

Using the notation $f_{Q_j} = \frac{1}{|Q_j|} \int_{Q_j} f(x) \, dx$, we write $f = g + b$ where $g$, the “good part”, is given by

$$g(x) = \begin{cases} f(x) & x \in \mathbb{R}^n \setminus \Omega \\ f_{Q_j} & x \in Q_j \end{cases}$$

Observe that $|g(x)| \leq 2^n \lambda$ a.e.
The “bad part” can be split as \( b = \sum_j b_j \), where \( b_j(x) = (f(x) - f_{Q_j})\chi_{Q_j}(x) \).

Let \( \tilde{Q}_j = 2Q_j \) and \( \tilde{\Omega} = \bigcup_j \tilde{Q}_j \).

We have

\[
\begin{align*}
&w(\{ y \in \mathbb{R}^n : |Tf(y)| > \lambda/2 \}) \\
&\leq w(\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : |Tg(y)| > \lambda/2 \}) + 2w(\tilde{\Omega}) + w(\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : |Tb(y)| > \lambda/2 \}).
\end{align*}
\]

Pick any \( p > 1 \) such that \( 1 < p < 1 + \epsilon \). Then, it follows that

\[
A_\epsilon = t \log |1 + t| \satisfies condition \int_\epsilon^\infty \left( \frac{t}{A_\epsilon(t)} \right)^{p'-1} \frac{dt}{t} < \infty,
\]

for some \( c > 0 \). Thus, we can apply Theorem 1.5 with this \( p \) to the first term, together with the fact that \( |g(x)| \leq 2^n \lambda \) a.e. Then, using an idea from [1] p. 282

\[
w(\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : |Tg(y)| > \lambda/2 \}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |Tg(y)|^p w(y)dy \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |g(y)|^p M_A(w)dy \leq \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \Omega} |g(y)| M_A(w)dy + \frac{C}{\lambda} \int_{\Omega} |g(y)| M_A(w)dy = \frac{C}{\lambda} (I + II)
\]

Since \( I \leq \int_{\mathbb{R}^n} |f(y)| M_A(w)dy \) we only need to estimate \( II \):

\[
II \leq \frac{C}{\lambda} \sum_j \int_{Q_j} |f_{Q_j}| M_A(w)dy \leq \frac{C}{\lambda} \sum_j \int_{Q_j} |f(x)| \frac{1}{|Q_j|} \int_{Q_j} M_A(w)dy.
\]

We shall make use of the following fact: for arbitrary Young function \( A \), non-negative function \( w \) with \( M_A w(x) < \infty \) a.e., cube \( Q \), and \( R > 1 \) we have

\[
M_A(\chi_{\mathbb{R}^n \setminus RQ} w)(y) \approx M_A(\chi_{\mathbb{R}^n \setminus RQ} w)(z) \quad (26)
\]

for each \( y, z \in Q \). This is an observation whose proof follows exactly as for the case of the Hardy–Littlewood maximal operator \( M \), cf. for instance [5] p. 159.
Then,

\[ II \leq C \sum_j \int_{Q_j} |f(x)| \, dx \inf_{Q_j} M_{A_\epsilon}(w_{X^n \setminus 2Q_j}) \leq C \sum_j \int_{Q_j} |f(x)| M_{A_\epsilon} w(x) \, dx \]

\[ \leq C \int_{\mathbb{R}^n} |f(x)| M_{A_\epsilon} w(x) \, dx. \]

The second term is estimated as follows:

\[ w(\tilde{\Omega}) \leq C \sum_j \frac{w(\tilde{Q}_j)}{|Q_j|} \leq \]

\[ \frac{C}{\lambda} \sum_j \frac{w(\tilde{Q}_j)}{|Q_j|} \int_{Q_j} |f(x)| \, dx \leq \frac{C}{\lambda} \sum_j \int_{Q_j} |f(x)| M w(x) \, dx \leq \]

\[ \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| M w(x) \, dx. \]

To estimate the last term we use the inequality

\[ \int_{\mathbb{R}^n \setminus \tilde{Q}_j} |T b_j(y)| w(y) \, dy \leq C \int_{\mathbb{R}^n} b_j(y) M w(y) \, dy, \]

with \( C \) independent of \( b_j \), which can be found in Lemma 3.3, p. 413, of [5]. Now, using this estimate with \( w \) replaced by \( w_{X^n \setminus \tilde{Q}_j} \) we have

\[ w(\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : |T b(y)| > \lambda/2\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |T b(y)| w(y) \, dy \leq \]

\[ \frac{C}{\lambda} \sum_j \int_{\mathbb{R}^n \setminus \tilde{Q}_j} |T b_j(y)| w(y) \, dy \leq \frac{C}{\lambda} \sum_j \int_{\mathbb{R}^n} |b_j(y)| M (w_{X^n \setminus \tilde{Q}_j})(y) \, dy \leq \]

\[ \frac{C}{\lambda} \sum_j \int_{Q_j} |b(y)| M (w_{X^n \setminus \tilde{Q}_j})(y) \, dy. \]

Since \( b = f - g \) this is at most

\[ \frac{C}{\lambda} \sum_j \left( \int_{Q_j} |f(y)| M w(y) \, dy + \int_{Q_j} |g(y)| M (w_{X^n \setminus \tilde{Q}_j})(y) \, dy \right) = \frac{C}{\lambda} (A + B) \]
To conclude the proof of the theorem is clear that we only need to estimate $B$. However
\[
B = \sum_j \int_{Q_j} |f_{Q_j}| M(w \chi_{\mathbb{R}^n \setminus Q_j})(y) dy \leq 
\]
\[
\sum_j \int_{Q_j} |f(x)| dx \frac{1}{|Q_j|} \int_{Q_j} M(w \chi_{\mathbb{R}^n \setminus Q_j})(x) dx \leq 
\]
\[
\sum_j \int_{Q_j} |f(x)| dx \inf_{Q_j} M(w \chi_{\mathbb{R}^n \setminus 2Q_j}) \leq \sum_j \int_{Q_j} |f(x)| M(w \chi_{\mathbb{R}^n \setminus 2Q_j})(x) dx \leq 
\]
\[
C \int_{\mathbb{R}^n} |f(y)| Mw(y) dy
\]
Here we have used again that $M(\chi_{\mathbb{R}^n \setminus 2Q})(y) \approx M(\chi_{\mathbb{R}^n \setminus 2Q})(z)$ for each $y, z \in Q$.
This concludes the proof of the theorem since we always have that $Mw(x) \leq M_\Lambda w(x)$ for each Young function $A$ and for each $x$.

\[\square\]

**Proof of Theorem 1.1:**

Let us assume that $M^{[p]}w$ is finite almost everywhere, since otherwise (6) is trivial. Let $A$ be the Young function
\[
A(t) = t \log^{[p]}(1 + t).
\]
A simple computation shows that $A$ satisfies condition (9), which is the hypothesis of Theorem 1.5. Then, Theorem 1.1 will follow if we prove the pointwise inequality
\[
M_\Lambda w(x) \leq C M^{[p]}w(x). \tag{27}
\]
Recall that $M_\Lambda$ is defined by $M_\Lambda f(x) = \sup_{x \in Q} \|f\|_{A,Q}$, where
\[
\|f\|_{A,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|} \int_{Q} A\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1\}.
\]
Then, it is enough to prove that there is constant $C$ such that for each cube $Q$
\[
\|f\|_{A,Q} \leq C \frac{1}{|Q|} \int_{Q} M^{[p]}w(x) dx.
\]
By assumption, the right hand side average is finite, and by homogeneity we can assume that is equal to one. Then, by the definition of Luxemburg norm we need to prove
\[
\frac{1}{|Q|} \int_Q A(w(y)) \, dy = \frac{1}{|Q|} \int_Q w(y) \log^p(1 + w(y)) \, dy \leq C.
\]
But this is a consequence of iterating the following inequality of E.M. Stein [7]
\[
\int_Q w(y) \log^k(1 + w(y)) \, dy \leq C \int_Q Mw(y) \log^{k-1}(1 + Mw(y)) \, dy,
\]
with \(k = 1, 2, 3, \ldots\).

To conclude the proof of the theorem, we are left with showing that for arbitrary \(1 < p < \infty\), the inequality
\[
\int_{\mathbb{R}^n} |Tf(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M^{[p]} w(x) \, dx,
\]
is false in general. To prove this assertion we consider the Hilbert transform
\[
Hf(x) = pv \int_{\mathbb{R}} \frac{f(y)}{x-y} \, dy.
\]
Then, by duality (29) is equivalent to
\[
\int_{\mathbb{R}} |Hf(x)|^{p'} M^{[p]} w(x)^{1-p'} \, dx \leq C \int_{\mathbb{R}} |f(x)|^{p'} w(x)^{1-p'} \, dx.
\]
Let \(f = w = \chi_{(-1,1)}\). A standard computation shows that
\[
M^k f(x) \approx \frac{\log^{k-1}(1 + |x|)}{|x|}, \quad |x| \geq e
\]
for each \(k = 1, 2, 3, \ldots\). Then, we have
\[
\int_{\mathbb{R}} |Hf(x)|^{p'} M^{[p]} w(x)^{1-p'} \, dx \geq C \int_{x>e} \left( \frac{1}{x} \right)^{p'} \left( \frac{\log^{[p]-1}(x)}{x} \right)^{1-p'} \, dx \approx \int_{x>e} \log^{(p)[1]}(x) \frac{dx}{x} = \infty,
\]
since \((p-1)\log(1+p)+1 \geq 0\). However, the right hand side of (30) equals \(\int_{\mathbb{R}} f(y)dy = 2 < \infty\).

\[ \]

\[ \]

**Proof of Theorem 1.2:**

As above, we shall assume that \(M^2w\) is finite almost everywhere. For \(0 < \epsilon < 1\) set as before \(A_{\epsilon}(t) = t \log^\epsilon(1 + t)\). Then, the inequality

\[ \int_{Q} w(y) \log^\epsilon(1 + w(y)) dy \leq C \int_{Q} Mw(y) dy, \]

whose proof is analogue to that of (28) using that the derivative of \(A_{\epsilon}(t)\) is less than \(1/t\), implies exactly as in the proof of Theorem 1.1 that

\[ M_{A_{\epsilon}}w(x) \leq CM^2w(x). \]

This concludes the proof of Theorem 1.2.

\[ \]

**Proof of Theorem 1.4:** By an standard argument, it is enough to show that there is a constant \(C\) such that

\[ \int_{\mathbb{R}^n} |Ta(y)| w(y)dy \leq C \]

for each \(Mw\)-atom \(a\). To prove this, suppose that \(\text{supp}(a) \subset Q\) for some cube \(Q\). Then

\[ \int_{\mathbb{R}^n} |Ta(y)| w(y)dy = \int_{3Q} |Ta(y)| w(y)dy + \int_{\mathbb{R}^n \setminus 3Q} |Ta(y)| w(y)dy = I + II. \]

Now, II is majorized, as in the proof of Theorem 1.6, by using Lemma 3.3, p. 413 of [5]

\[ II \leq C \int_{\mathbb{R}^n} |a(y)| Mw(y)dy \leq \frac{C}{Mw(Q)} \int_{Q} Mw(y)dy = C, \]

where \(C\) is independent of \(a\).

For I we use the fact that any singular integral operator \(T : L^{\infty}(Q, \frac{dx}{|Q|}) \rightarrow L_{L_{\exp}}(Q, \frac{dx}{|Q|})\). Then

\[ I = |3Q| \left| \frac{1}{|3Q|} \int_{3Q} |Ta(y)| w(y)dy \right| \leq C|Q||Ta|_{L_{\exp,3Q}}||w|_{L_{\log,L,3Q}} \leq \]

\[ \]

\[ \]
\[ \leq C |Q| \|a\|_{\infty,3Q} \frac{1}{|3Q|} \int_{3Q} Mw(y)dy \leq C, \]

by (28) and by the definition of \( Mw \)-atom. This finishes the proof of Theorem 1.4. □

We shall end this section by disproving inequality

\[ w(\{ y \in \mathbb{R}^n : |Tf(y)| > \lambda \}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(y)|^p M[w]^p(y)dy \quad (31) \]

from remark 1.3, whenever \( p \) is greater than one but not an integer.

Consider \( T = H \) the Hilbert transform as above. For \( \lambda > 0 \), we let \( f = \chi_{(1,e^\lambda)} \), and \( w = \chi_{(0,1)} \). Then for \( y \neq 1, e^\lambda \)

\[ Hf(y) = \log \left| \frac{y-1}{y-e^\lambda} \right|. \]

When \( y \in (0,1) \) we have

\[ |Hf(y)| = |\log \left| \frac{y-1}{y-e^\lambda} \right| | = \log \frac{e^\lambda - y}{1 - y} > \log e^\lambda = \lambda. \]

Then, assuming that (31) holds for all \( \lambda \) we had

\[ 1 = \int_0^1 w(y)dy \leq w(\{ y \in (0,1) : |Hf(y)| > \lambda \}) \leq \]

\[ \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(y)|^p M[w]^p(y)dy = \frac{C}{\lambda^p} \int_1^{e^\lambda} M[w]^p(y)dy \approx \]

\[ \approx \frac{1}{\lambda^p} \int_1^{e^\lambda} \log |y-1| w(y)dy \approx \lambda^{[p]-p}. \]

By letting \( \lambda \to \infty \) we see that this a contradiction when \( p \) is not an integer.

There is another argument due to S. Hofmann, and is as follows. Since \( p \) is not an integer we can find an small \( \epsilon > 0 \) such that \( [p] < p - \epsilon < p < p + \epsilon < [p] + 1 \).

Then, (31) implies that \( M \) is at once of weak type \( (p-\epsilon,p-\epsilon) \) and \( (p+\epsilon,p+\epsilon) \) with respect to the weights \( (w, M[w]) \). Then, by the Marcinkiewicz interpolation theorem \( M \) is of strong type \( (p,p) \) with respect to the weights \( (w, M[w]) \). But this is a contradiction as shown in Theorem 1.1.
3 Calderón–Zygmund operators

In this section we shall state our main results for the more general Calderón–Zygmund operators.

We recall the definition of a Calderón–Zygmund operator in \( \mathbb{R}^n \). A kernel on \( \mathbb{R}^n \times \mathbb{R}^n \) will be a locally integrable complex–valued function \( K \), defined on \( \Omega = \mathbb{R}^n \times \mathbb{R}^n \setminus \text{diagonal} \). A kernel \( K \) on \( \mathbb{R}^n \) satisfies the standard estimates, if there exist \( \delta > 0 \) and \( C < \infty \) such that for all distinct \( x, y \in \mathbb{R}^n \) and all \( z \) such that \( |x - z| < |x - y|/2 \):

(i) \( |K(x, y)| \leq C |x - y|^{-n} \);
(ii) \( |K(x, y) - K(z, y)| \leq C \left( \frac{|x - z|}{|x - y|} \right)^{\delta} |x - y|^{-n} \);
(iii) \( |K(y, x) - K(y, z)| \leq C \left( \frac{|x - z|}{|x - y|} \right)^{\delta} |x - y|^{-n} \).

We say that a linear and continuous operator \( T : C_0^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n) \) is associated with a kernel \( K \), if

\[
\langle Tf, g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y)g(x)f(y) \, dx \, dy,
\]

whenever \( f, g \in C_0^\infty(\mathbb{R}^n) \) with \( \text{supp}(f) \cap \text{supp}(g) = \emptyset \).

We say that \( T \) is a Calderón–Zygmund operator if the associated kernel \( K \) satisfies the standard estimates, and if it extends to a bounded linear operator in \( L^2(\mathbb{R}^n) \).

Theorem 3.1: Let \( 1 < p < \infty \), and let \( T \) be a Calderón–Zygmund operator. Then, there exists a constant \( C \) such that for each weight \( w \)

\[
\int_{\mathbb{R}^n} |Tf(y)|^p w(y) \, dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M^{|p|+1} w(y) \, dy,
\]

and there exists another constant \( C \) such that for all \( \lambda > 0 \)

\[
w(\{ y \in \mathbb{R}^n : |Tf(y)| > \lambda \}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)|^p M^2 w(y) \, dy.
\]

The proof of Theorem 3.1 is essentially the same as Theorems 1.1 and 1.2, after observing that the adjoint \( T^* \) of any Calderón–Zygmund operator \( T \) is also a Calderón–Zygmund operator with kernel \( K^*(x, y) = K(y, x) \).
There are corresponding results to Theorems 1.2, 1.4, 1.5, and for 1.6 for any Calderón–Zygmund operator. We shall omit the obvious statements.

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References


