CALDERÓN–ZYGUMUND THEORY RELATED TO POINCARÉ-SHOBOLEV INEQUALITIES, FRACTIONAL INTEGRALS AND SINGULAR INTEGRAL OPERATORS

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Paseky’s lecture notes
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Chapter 1

A brief introduction to the $A_p$ theory of weights

The Hardy-Littlewood maximal function is the operator defined by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_{Q} |f(y)| \, dy$$

where the supremum is taken over all the cubes containing $x$. By cube we always mean a cube with sides parallel to the axes. There are variations of this definition such as replacing cubes by balls or just considering cubes or balls centered at $x$ but all of them are equivalent with dimensional constants.

Maximal functions arise very naturally in analysis, for proving theorems about the existence almost everywhere of limits, for controlling pointwise important objects such as the Poisson Integrals or for controlling, not pointwise but at least in average, other basic operators such as singular integral operators.

The model example of existence almost everywhere of limits is the Lebesgue differentiation theorem:

$$f(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \, dy$$

Indeed, as it is well known this follows from the following “weak type” estimate:

**Theorem 1.0.1 (Hardy–Littlewood–Wiener)** There exists a finite constant $C$ such that for each positive $\lambda$ the following inequality holds,

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda \}| \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \, dx$$

As a consequence of (1.1) we can derive:

**Corollary 1.0.2** Let $1 < p < \infty$ then there exists a constant $C$ such that

$$\int_{\mathbb{R}^n} (Mf)^p \, dx \leq C \int_{\mathbb{R}^n} |f|^p \, dx$$

(1.2)
We say that $w$ is a weight if $w$ is a a.e. positive locally integrable function in $\mathbb{R}^n$. If $E$ is any measurable set we denote $w(E) = \int_E w$.

**Theorem 1.0.3**  

a) There exists a constant $C$ such that for all $\lambda$ and $f$

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda \}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| Mw(x) dx. \tag{1.3}$$

b) As a consequence if $1 < p < \infty$ there exists a constant $C$ such that for all $f$

$$\int_{\mathbb{R}^n} (Mf)^p w(x) dx \leq C \int_{\mathbb{R}^n} |f|^p Mw(x) dx. \tag{1.4}$$

We may think of this inequality as a kind of duality for the (nonlinear) operator. It can also be seen as an antecedent of the $A_p$ theory weights. It proved by C. Fefferman and E. Stein in [FS] to derive a vector–valued extension of (1.0.2). Indeed, for a vector of functions $f = \{f_i\}$ and for $q > 0$ the operator $M_q$ is defined by

$$M_qf(x) = \left( \sum_{i=1}^{\infty} (Mf_i(x))^q \right)^{1/q}.$$ 

This operator can also be seen as a generalization of the classical integral of Marcinkiewicz. Then if we denote $|f(x)|_q = (\sum_{i=1}^{\infty} |f_i(x)|^q)^{1/q} = \|f(x)\|_{\ell^q}$ we have for $1 < p < \infty$ and $1 < q \leq \infty$

$$\int_{\mathbb{R}^n} M_qf(x)^p dx \leq C \int_{\mathbb{R}^n} |f(x)|_q^p dx. \tag{1.5}$$

A corresponding weak type $(1,1)$ also holds in the range $1 < p < \infty$. These estimates play an important role in Harmonic Analysis, specially in the theory of Littlewood-Paley. To some extent these operators behave more as a singular integral (cf. Chapter 6) than as a maximal function. This is very well reflected in the work [RRT] where $M_qf$, as well many other operators, is seen as a vector–valued singular integral operator.

**Proof:** The proof of b) follows from the Marcinkiewicz interpolation theorem since $M : L^\infty(u) \to L^\infty(v)$ for arbitrary weights $u$ and $v$. (see also the proof of (4.6) in Lemma 4.2.2).

The proof of a) is based on the classical Vitali covering lemma:

**Lemma 1.0.4** (Vitali) Let $\mathcal{F} = \{Q_i\}_{i=1,\ldots,N}$ be a finite family of cubes (or balls) in $\mathbb{R}^n$. Then we can extract from $\mathcal{F}$ a sequence of pairwise disjoint cubes $\mathcal{F}' = \{Q_j\}_{j=1,\ldots,M}$ such that

$$\bigcup_{Q \in \mathcal{F}} Q \subset \bigcup_{j=1}^m 5Q_j.$$
See [Ma] p.23.

Now if we let $\Omega_{\lambda} = \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$ for a given $\lambda$ and let $K$ be any compact subset contained in $\Omega_{\lambda}$. Let $x \in K$ then by definition of the maximal function there is a cube $Q = Q_x$ containing $x$ such that

$$\frac{1}{|Q|} \int_Q |f(y)| dy > \lambda. \quad (1.6)$$

Then $K \subset \bigcup_{x \in K} Q_x$ and by compactness we can extract a finite family of cubes $\mathcal{F} = \{Q\}$ such that $K \subset \bigcup_{Q \in \mathcal{F}} Q$ and where each cube satisfies (1.6). Then by the Vitali lemma we can extract a pairwise disjoint family of cubes $\{Q_j\}_{j=1}^M$ such that $\bigcup_{Q \in \mathcal{F}} Q \subset \bigcup_{j=1}^M 5Q_j$. Then

$$w(K) \leq \sum_{j=1}^M w(5Q_j) \leq \frac{1}{\lambda} \sum_{j=1}^M w(5Q_j) \frac{1}{|Q_j|} \int_{Q_j} |f(y)| dy$$

$$\leq \frac{C}{\lambda} \sum_{j=1}^M \frac{w(5Q_j)}{|5Q_j|} \int_{Q_j} |f(y)| dy \leq \frac{C}{\lambda} \sum_{j=1}^M \int_{Q_j} |f(y)| Mw(y) dy$$

$$\leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| Mw(y) dy. \quad \Box$$

The main breakthrough in the theory arrived in the early seventies with the discovery by B. Muckenhoupt in [Mu] of the right class of weights for which the Hardy-Littlewood maximal function is bounded on $L^p(w)$, $p > 1$. This result opened up the possibility of studying operators with higher singularity.

**Definition 1.0.5** A weight $w$ belongs to the class $A_p$, $1 < p < \infty$, if there is a constant $K$ such that

$$\left( \frac{1}{|Q|} \int_Q w(y) \, dy \right)^{1/p} \left( \frac{1}{|Q|} \int_Q w(y)^{1-p'} \, dy \right)^{p-1} \leq K \quad (1.7)$$

for each cube $Q$. A weight $w$ belongs to the class $A_1$ if there is a constant $K$ such that

$$\frac{1}{|Q|} \int_Q w(y) \, dy \leq K \inf_Q w. \quad (1.8)$$

We will denote the infimum of the constants $K$ by $[w]_{A_p}$.

Observe that $[w]_{A_p} \geq 1$ by Jensen’s inequality.

Sometimes it is much more convenient to use the following equivalent condition for the $A_1$ weight: there is a constant $K$ such that for each $x$

$$Mw(x) \leq K w(x)$$
Observe that it follows from inequality 1.3 that if \( w \) is an \( A_1 \) weight then
\[
 w(\{ x \in \mathbb{R}^n : Mf(x) > \lambda \}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \, w(x) \, dx 
\] (1.9)

It is not hard to see that this condition is also equivalent to saying that \( w \in A_1 \).

**Theorem 1.0.6 (Muckenhoupt)** Let \( p > 1 \). The following conditions are equivalent:

a) \( w \in A_p; \)

b) there exists a constant \( K \) such that for each cube \( Q \) and nonnegative function \( f \)
\[
 \left( \frac{1}{|Q|} \int_Q f(y) \, dy \right)^p \, w(Q) \leq K \int_Q f(y)^p \, w(y) \, dy 
\]

c) \( M : L^p(w) \to L^{p,\infty}(w) \)

d) \( M : L^p(w) \to L^p(w) \)

Conditions a), b) and c) are also equivalent when \( p = 1 \).

The main example of \( A_1 \) weights are provided by the following lemma of Coifman and Rochberg (cf. [GCRdF] p. 158).

**Lemma 1.0.7** Let \( \mu \) be a positive Borel measure such that \( M\mu(x) < \infty \), a.e. \( x \in \mathbb{R}^n \). Then for each \( 0 \leq \delta < 1 \) the function \( (M\mu)^\delta \in A_1 \)

In fact it is also shown in that paper that any \( A_1 \) weight \( w \) can be written essentially of this form. Furthermore, it is possible to show that
\[
 w \in A_p \text{ if and only if } w = w_1^{1-p} w_2 
\]

where \( w_1 \) and \( w_2 \) are \( A_1 \) weights. That the last condition was a \( A_p \) was already known by B. Muckenhoupt, but it was P. Jones who proved the necessity of the factorization theorem with a very difficult proof. The modern approach uses completely different arguments. It is due to J. L. Rubio de Francia and it is presented in [GCRdF] where we remit the reader for more information about the \( A_p \) theory of weights.

**Definition 1.0.8** The \( A_\infty \) class of weights is defined in a natural way by
\[
 A_\infty = \cup_{p>1} A_p. 
\]
The $A_\infty$ class of weights shares a lot of interesting properties. We present in Appendix 8 several characterizations. Some of them will be used along these notes.

Throughout these notes all notation is standard or will be defined as needed. All cubes are assumed to have their sides parallel to the co-ordinate axes. Given a cube $Q$, $l(Q)$ will denote the length of its sides and for any $r > 0$, $rQ$ will denote the cube with the same center as $Q$ and such that $l(rQ) = rl(Q)$. We will denote the collection of all dyadic cubes by $D$ and by $D(Q)$ to the collection of all dyadic cubes relative to the (non necessarily dyadic) cube $Q$. $Q(x,r)$ will denote the cube centered at $x$ and with side–length $2r$. By weights we will always mean a.e. positive, locally integrable functions which are positive on a set of positive measure. Given a Lebesgue measurable set $E$ and a weight $w$, $|E|$ will denote the Lebesgue measure of $E$ and $w(E) = \int_E w \, dx$. Given $1 < p < \infty$, $p' = p/(p - 1)$ will denote the conjugate exponent of $p$. Finally, $C$ will denote a positive constant whose value may change at each appearance.
Chapter 2

\[ L^p \] properties of functions with controlled oscillation

For a locally integrable function \( f \) we define the oscillation of \( f \) over a cube \( Q \) as the number

\[
Osc(Q, f) = \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy \approx \inf_{c \in \mathbb{R}} \frac{1}{|Q|} \int_Q |f(y) - c| dy
\]

\[
\approx \frac{1}{|Q|^2} \int_Q \int_Q |f(y) - f(x)| dy dx.
\]

\( Osc(Q, f) \) can be seen as the modern way of checking how smooth is a function. For instance, this can be seen in the characterization (2.1) given below of the Hölder-Lipschitz spaces and also when considering functions in the Sobolev class, see for instance (2.3) and (2.4). But it goes beyond since R. Coifman and R. Rochberg proved in [CR] that maximal functions themselves have some sort of smoothness. They showed that \( Osc(Q, \log M \mu) \) is bounded by a dimensional constant provided \( M \mu \) is finite almost everywhere. What this means is that there is a cancellation phenomenon undergoing. Indeed, this result would not make sense if we remove the average \( f_Q \) in the definition of \( Osc(Q, f) \).

In this chapter we will study properties of classes of functions which have some control on the oscillation. In particular we will emphasize on \( L^p \) self-improving properties. Roughly speaking we will show that higher integrability properties of functions belonging to \( B.M.O. \) (or Hölder-Lipschitz) and the higher integrability of Sobolev functions are essentially the same phenomenon.

The results and techniques we present in this chapter are taken from [FPW1] and [MP1]. These papers are specially motivated by the work P. Hajlasz and P. Koskela [HaK1], whose roots got back to the theory of Quasiconformal Mappings as can be seen in [HK], but also by the work of L. Saloff–Coste in [SC].
2.1 BMO and Hölder-Lipschitz spaces

Recall that a function $f$ is said to satisfy the Hölder condition with exponent $\alpha$, $0 < \alpha < 1$, if there exists a constant $C$ such that for each $x$ and $y$

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}.$$ 

When $\alpha = 1$ the function is said to satisfy the Lipschitz condition. We will denote them by $\Lambda(\alpha)$ and it is well–known how relevant is this condition in Analysis. The main examples of functions in $\Lambda(\alpha)$ are given by $f(x) = |x|^\alpha$, $0 < \alpha \leq 1$.

From our point view it is more interesting the following characterization due to Campanato and Morrey:

Let $0 < \alpha \leq 1$, then $f \in \Lambda(\alpha)$ if and only if

$$\frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy \leq C\ell(Q)^\alpha,$$ 

(2.1)

where $C$ is a constant independent of the cube $Q$.

Observe that $\Lambda(0)$ does not yield any interesting space, however if we replace in (2.1) $\alpha$ by zero we obtain the celebrated Bounded Mean Oscillation space introduced by F. John and L. Nirenberg in the early sixties in their study of partial differential equations. It is usually denoted by $B.M.O.$: $f \in B.M.O.$ if there exists a constant $C$ such that for each cube $Q$

$$\frac{1}{|Q|} \int_Q |f - f_Q| \leq C.$$ 

The main example is $\log M(\mu)(x)$. If we take $\mu$ to be the point mass at the origin we get $\log |x|$ which is the classical example of unbounded $B.M.O.$ function.

In any case they share the following self–improving property:

Let $0 \leq \alpha \leq 1$ and let $f$ be a function satisfying (2.1). Then for each $1 < p < \infty$ there exists a constant $c$ such that

$$\left( \frac{1}{|Q|} \int_Q |f - f_Q|^p \right)^{1/p} \leq c\ell(Q)^\alpha.$$ 

(2.2)

2.2 Poincaré–Sobolev Inequalities

Poincaré–Sobolev inequalities play a main role in many places in Analysis. They are crucial tools in many important results for Partial Differential Equations, such as the De Giorgi-Nash-Moser theorem, Harnack's inequality, properties of the Green functions for second order elliptic equations and existence results for semilinear equations (cf. for
instance some recent books such as [HKM] or citeMaly. It is also important in the theory of Quasiconformal mappings as can be seen in the recent theory developed by J. Heinonen and P. Koskela [HK]. Very roughly, this theory shows that if $X$ is a proper metric space, i.e. an space where closed balls are compact, endowed with a mesure which is Ahlfors–David regular (hence doubling) then we can develop an analogue of the classical theory of Quasiconformal mappings if the space ‘admits’, in an appropriate sense, a $(1,p)$ type Poincaré inequality.

• The model example is the classical $(1,1)$ Poincaré inequality:

$$\frac{1}{|Q|} \int_Q |f - f_Q| \leq C \frac{\ell(Q)}{|Q|} \int_Q |\nabla f|,$$ (2.3)

where $C$ a constant independent of both $Q$ and $f$. In fact there is a more general version of this estimate for $p \geq 1$

$$\left(\frac{1}{|Q|} \int_Q |f - f_Q|^p\right)^{1/p} \leq C \ell(Q) \left(\frac{1}{|Q|} \int_Q |\nabla f|^p\right)^{1/p}. \quad (2.4)$$

In fact we will show in Lemma 2.3.1 that they are both equivalent as a first step to a more general result in Theorem 2.4.2

An important observation is that this inequality is not only uniform on cubes but also on an appropriate class of functions, which may be a Sobolev class of functions or the Lipschitz class since such functions are differentiable almost everywhere by the Rademacher–Stepanov theorem.

• A second example is a sharp version of inequality (2.3) which turns to be equivalent to the isoperimetric inequality: there exists a constant $C$ such that

$$\left(\frac{1}{|Q|} \int_Q |f - f_Q|^n\right)^{1/n'} \leq C \frac{\ell(Q)}{|Q|} \int_Q |\nabla f|.$$

The version corresponding to $p \geq 1$ is

$$\left(\frac{1}{|Q|} \int_Q |f - f_Q|^{p^*}\right)^{1/p^*} \leq C \ell(Q) \left(\frac{1}{|Q|} \int_Q |\nabla f|^p\right)^{1/p}. \quad (2.6)$$

Here $p^* = \frac{pm}{n-p}$ denotes the Sobolev exponent. Observe that $1^* = \frac{n}{n-1} = n'$ is the exponent from the isoperimetric inequality.

• We will show below that indeed both inequalities (2.4) and (2.6) are equivalent. In fact we will stress this relationship by introducing weights in these inequalities. These weighted Poincaré inequalities are basic in the study of the smoothness of the solutions of degenerate elliptic equation (see [FKS] and [HKM]).
The main example holds for any $w \in A_2$:

$$
\left( \frac{1}{w(Q)} \int_Q |f - f_Q|^2 w \right)^{1/2} \leq C \ell(Q) \left( \frac{1}{w(Q)} \int_Q |
abla f|^2 w \right)^{1/2}. \tag{2.7}
$$

To run the Moser iterative technique the following was derived in [FKS]: there exists a tiny positive constant $\epsilon$ for which

$$
\left( \frac{1}{w(Q)} \int_Q |f - f_Q|^{2+\epsilon} w \right)^{1/(2+\epsilon)} \leq C \ell(Q) \left( \frac{1}{w(Q)} \int_Q |
abla f|^2 w \right)^{1/2}. \tag{2.8}
$$

We will show below that these examples are particular cases of a more general situation.

### 2.3 Fractional Integrals and Poincaré inequalities

It is well known that there is an intimate relationship between smoothness and fractional integration. This is best reflected in the following estimate: let $f$ be a, say $C^1$ function, on a cube $Q$, then there is a universal constant $C$ such that for each $x \in Q$

$$
|f(x) - f_Q| \leq C I_1(|\nabla f|\chi_Q)(x). \tag{2.9}
$$

Here $I_\alpha$, $0 < \alpha < n$, denotes the fractional integral or Riesz potentials of order $\alpha$ and it is defined by

$$
I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy.
$$

We will prove below the well–known fact that (2.9) implies (2.3). However what it is new is that in fact these two estimates are equivalent. In fact there is no need to work with the gradient as the following characterization shows.

**Lemma 2.3.1** Let $\alpha > 0$, $f$ a locally integrable function and let $g$ be a nonnegative function. Then the following conditions are equivalent:

a) There exists a constant $C$ such that for all cubes $Q$:

$$
\frac{1}{|Q|} \int_Q |f - f_Q| \leq C \frac{\ell(Q)^\alpha}{|Q|} \int_Q g.
$$

b) There exists a constant $C$ such that for all cubes $Q$:

$$
|f(x) - f_Q| \leq C I_\alpha(g\chi_Q)(x) \quad a.e. \ x \in Q \tag{2.10}
$$
c) there is a positive constant $C$ such that for every weight $w$ and for all cubes $Q$:

$$
\frac{1}{|Q|} \int_Q |f - f_Q| w \leq C \frac{\ell(Q)^\alpha}{|Q|} \int_Q g(M(w)\chi_Q)
$$

\[ \text{d) Let } p \geq 1, \text{ then there exists a constant } C \text{ such that for all cubes } Q:
\]

$$
\left( \frac{1}{|Q|} \int_Q |f - f_Q|^p \right)^{1/p} \leq C \ell(Q)^\alpha \left( \frac{1}{|Q|} \int_Q g^p \right)^{1/p}.
$$

The key part is the classical integral representation formula of Sobolev type (2.10). This formula is also available in different and more general contexts thanks to the work of [FLW2] which we adapt to our situation.

\textbf{Proof: } a) \Rightarrow b) \text{ It is in this implication where we adapt the ideas from following [FLW2]. Fix } Q = Q_0 \text{ and let } x \in Q \text{ and consider the unique one-way infinite chain of dyadic cubes } \{Q_k\}_{k=0}^\infty, \text{ } Q_{k+1} \subset Q_k \text{ such that }

$$
\{x\} = \bigcap_{k=0}^\infty Q_k.
$$

See Appendix I. For each $k$ we let $f_k = \frac{1}{|Q_k|} \int_{Q_k} f$. Then by the Lebesgue differentiation theorem we have a.e. $x \in Q$ that

$$
|f(x) - f_Q| = |\lim_{k \to \infty} f_k - f_Q| = \sum_{k=0}^\infty |f_k - f_{k+1}|
$$

\[ \leq \sum_{k=0}^\infty \frac{1}{|Q_{k+1}|} \int_{Q_{k+1}} |f - f_{Q_k}| \leq 2^n \sum_{k=0}^\infty \frac{1}{|Q_k|} \int_{Q_k} |f - f_{Q_k}|.
\]

$$
\leq C \sum_{k=0}^\infty \frac{\ell(Q_k)^\alpha}{|Q_k|} \int_{Q_k} g(y) \sum_{k=0}^\infty \frac{\ell(Q_k)^\alpha}{|Q_k|} \chi_{Q_k}(y) \, dy.
$$

We now use that $Q_k \subset Q(x, \ell(Q_k))$. Observe that the natural estimate

$$
\frac{\ell(Q_k)^\alpha}{|Q_k|} \chi_{Q_k}(y) \leq \frac{\ell(Q_k)^\alpha}{|Q_k|} \chi_{Q(x, \ell(Q_k))}(y) \leq |y - x|^{-n} \chi_{Q(x, \ell(Q_k))}(y)
$$

is not sufficient to sum up the series, but we can proceed as follows: let $\epsilon$ be such that $0 < \epsilon < n - \alpha$, then

$$
\sum_{k=0}^\infty \frac{\ell(Q_k)^\alpha}{|Q_k|} \chi_{Q_k}(y) = \sum_{k=0}^\infty \frac{c}{\ell(Q_k)^{\alpha - \epsilon}} \frac{1}{\ell(Q_k)^\epsilon} \chi_{Q_k}(y) \leq \frac{c}{|y - x|^{n - \alpha - \epsilon}} \sum_{k=0}^\infty \frac{1}{\ell(Q_k)^\epsilon} \chi_{Q_k}(y),
$$

since $x, y \in Q_k$. Now pick the integer $k_0$ such that $2^{k_0} \approx \frac{\ell(Q)}{|y - x|}$. Then

$$
\frac{c}{|y - x|^{n - \alpha - \epsilon}} \sum_{k=0}^\infty \frac{1}{\ell(Q_k)^\epsilon} \chi_{Q_k}(y) = \frac{c}{|y - x|^{n - \alpha - \epsilon}} \frac{1}{(2^{k_0})} \ell(1 + 2^\epsilon + 2^{2\epsilon} + \ldots + 2^{k_0\epsilon}) \approx \frac{1}{|y - x|^{n - \alpha}}.
$$
this shows that a.e. for \( x \in Q \)
\[
|f(x) - f_Q| \leq C_{\varepsilon,n} \int_Q \frac{g(y)}{|y - x|^{n-\alpha}} dy.
\]

\( b) \Rightarrow c) \) Indeed by assumption and Fubini’s
\[
\int_Q |f(x) - f_Q| w(x) dx \leq c \int_Q \int_Q \frac{g(y)}{|x - y|^{n-\alpha}} dy w(x) dx = \int_Q \int_Q \frac{w(x)}{|x - y|^{n-\alpha}} dx g(y) dy
\]
For the inner integral we do the following: if the cube \( Q = Q(z, r) \) is centered at \( z \) and with radius \( r \) we have for each \( y \in Q \) that \( Q \subset Q(y, 2r) \). Then if we let \( r_0 = 2r \), \( r_k = r_{k-1}/2 \), and \( A(y, k) = Q(y, r_k) \setminus Q(y, r_{k-1}) \), \( k = 1, 2, \ldots \) we have
\[
\int_{Q(z, r)} \frac{w(x)}{|x - y|^{n-\alpha}} dx \leq \int_{Q(y, 2r)} \frac{w(x)}{|x - y|^{n-\alpha}} dx = \sum_{k=1}^{\infty} \int_{A(y, k)} \frac{w(x)}{|x - y|^{n-\alpha}} dx
\]
\[
\leq C \sum_{k=0}^{\infty} \frac{r_k^\alpha}{r_k^n} \int_{Q(y, r_k)} w(x) dx \leq C M w(y) \sum_{k=1}^{\infty} \frac{r_k^\alpha}{2k^\alpha} = C \ell(Q)^\alpha M w(y)
\]

\( c) \Rightarrow d) \) The case \( p = 1 \) follows by taking \( w \equiv 1 \). The case \( p > 1 \) follows by a duality argument combined with a good weighted estimate. This is a very fruitful idea as can be seen in the following Chapters. Indeed, we can write
\[
\left( \int_Q |f - f_Q|^p \right)^{1/p} = \sup \{ |\int_Q (f - f_Q) h| \}
\]
where the supremum is taken over all functions \( h \) such that \( \int_Q |h|^{p'} = 1 \). Taking one of these \( h \)’s we have by hypothesis that
\[
|\int_Q (f - f_Q) h| \leq \int_Q |f - f_Q||h| \leq C \ell(Q)^\alpha \int_Q g M(h \chi_Q)
\]
\[
\leq C \ell(Q)^\alpha \left( \int_Q g^p \right)^{1/p} \left( \int_Q (M(h \chi_Q))^p \right)^{1/p'}
\]
\[
\leq C \ell(Q)^\alpha \left( \int_Q g^p \right)^{1/p} \left( \int_Q |h|^{p'} \right)^{1/p'} = C \ell(Q)^\alpha \left( \int_Q g^p \right)^{1/p}
\]
\( d) \Rightarrow a) \) Just take \( p = 1 \)
\( \square \)
2.4 \textit{$L^p$ self–improving property}

In this section we will show that Lemma 2.3.1 can be pushed further to get higher $L^p$ integrability. The methods we will use are completely different and more related to the Calderón-Zygmund theory. We will be working in a more general framework. We consider “functionals” $a$

$$a : \mathcal{Q} \to (0, \infty)$$

$\mathcal{Q}$ denotes the family of all cubes from $\mathbb{R}^n$. Our model example is associated to the fractional average

$$a(Q) = \frac{\ell(Q)^\alpha}{|Q|} \int_Q d\nu,$$  \hspace{1cm} (2.11)

where $0 \leq \alpha$ and $\nu$ is a nonnegative measure. Some interesting cases are provided when the measure $\nu$ is absolutely continuous, with $d\nu = g \, dx$, with $g \in A_\infty$ as we shall see below (for instance a polynomial weight). Motivated by the theory developed in [HK] we can consider a more general case

$$a(Q) = \ell(Q)^\alpha \left( \frac{\nu(\lambda Q)}{|Q|} \right)^{1/p},$$  \hspace{1cm} (2.12)

with $\lambda \geq 1$. Observe that these functionals are not necessarily radial, i.e., $a$ need not be of the form $a(Q) = \varphi(|Q|)$ where $\varphi(0, \infty) \to (0, \infty)$ which under certain restrictions on $\varphi$ have appeared in the literature as generalizations of the Lipschitz spaces [Ja].

Now let $f$ be a locally integrable function on $\mathbb{R}^n$ and let $a$ be a functional. Assume that $f$ satisfies the following estimate

$$\frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy \leq a(Q)$$  \hspace{1cm} (2.13)

uniformly on $Q \in \mathcal{Q}$. Our goal is to find a condition on $a$ such that (2.13) implies a $L^q$ self–improving property of the form

$$\left( \frac{1}{|Q|} \int_Q |f(y) - f_Q|^q \, dy \right)^{1/q} \leq C a(Q).$$  \hspace{1cm} (2.14)

We impose the following discrete condition on the functional $a$ relative to a locally integrable weight function $w$.

To get such an improvement we must impose a mild discrete type condition which reflects the underlying geometry of the space. We expect in the first case that the exponent $q$ will lie on a certain generally open range $(1, r)$ where $r = r(a)$ is an exponent which depends upon a mild condition on the functional $a$. To be more precise we shall assume the following condition.

\textbf{Definition 2.4.1} Let $0 \leq r < \infty$ and let $w$ be a weight. We say that the functional $a$ satisfies the (weighted) $D_r$ condition if there exists a finite constant $c$ such that for each
cube $Q$ and any family $\Delta$ of pairwise disjoint dyadic subcubes of $Q$,
\[ \sum_{P \in \Delta} a(P)^r w(P) \leq C^r a(Q)^r w(Q). \] (2.15)

Observe that the condition is of local nature and it reminds a bit the Carleson condition. It is convenient to denote the best constant $C$ by $\|a\|$. Observe that $\|a\| \geq 1$ and that by Hölder’s inequality, the family $\{D_r\}$ is decreasing as $r$ increases, that is, if $r < s$, $D_s \subset D_r$. Hence for a functional $a \in D_r$ we can define the optimal exponent to which it belongs, namely
\[ r_a = \sup \{r : a \in D_r\}. \]

For instance if we consider the fractional functional
\[ a(Q) = \frac{\ell(Q)^\alpha}{|Q|^\alpha} \int_Q d\nu, \]
we have that $r_a = \frac{n}{n-\alpha}$ and furthermore $a$ satisfies the (unweighted) $D_{r_a}$. Indeed if $\Delta$ is an arbitrary family of disjoint cubes we have:
\[ \sum_{P \in \Delta} a(P)^\frac{n}{n-\alpha} |P| = \sum_{P \in \Delta} \left( \int_P d\nu(y) \right)^\frac{n}{n-\alpha} \leq \left( \sum_{P \in \Delta} \int_P d\nu(y) \right)^\frac{n}{n-\alpha} \]
\[ \leq \left( \int_Q d\nu(y) \right)^\frac{n}{n-\alpha} = a(Q)^\frac{n}{n-\alpha} |Q| \] (2.16)

If we consider instead the functional (2.12) with $\lambda > 1$ it is possible to show (see [FPW1]) that with $\alpha > 0, 1 \leq p < n/\alpha$ then $a \in D_t$ for $1 < t < pn/(n - \alpha p)$ and hence $r_a = pn/(n - \alpha p)$. However, it does not satisfy $D_{r_a}$. On the other hand this is good enough to recover the strong type estimate in part b) of Theorem 2.4.2, but not part a), the optimal weak type endpoint estimate. Observe that $pn/(n - \alpha p)$ is a more generally Sobolev exponent involving the index $\alpha$.

It can also be shown that for particular $\nu$ a is not in $D_{\frac{n}{n-\alpha} + \epsilon}$ for all $\epsilon > 0$ for a general measure $\nu$. Last observation shows that the class $D_r$ does not share in general a self–improving or openness property of the sort
\[ D_r \Rightarrow D_{r+\epsilon}, \]
which may be of interest to get optimal endpoint estimates (see Theorem 2.4.2). In some cases this openness property holds. For instance consider
\[ a(Q) = \frac{|Q|^\alpha/n}{|Q|} \int_Q w(y) \, dy, \] (2.17)
with $w \in A_\infty$. Then one can shows that $a \in D_{\frac{n}{n-\alpha} + \epsilon}$ where $\epsilon$ depends upon the $A_\infty$ constant of $w$. 

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On the other hand side there are examples of functionals which satisfy the $D_r$ condition for all $r$. Indeed, this the case of any nondecreasing nonradial case, namely: $a(P) \leq a(Q)$, $P \subset Q$. In fact it belongs to $\bigcap_{r>1} D_r$. An example of this situation is given by

$$a(Q) = \frac{1}{|Q|} \int_Q |\pi(y)| \, dy,$$  \hspace{1cm} (2.18)

where $\pi$ is a polynomial or more generality any weight satisfying the reverse Hölder property of order infinity.

Nondecreasing functionals are special as can be seen in Appendix III. Indeed if $f$ satisfies (2.13) with nondecreasing $a$ then $f$ is at least of exponential type including the John and Nirenberg classical theorem. In fact it is shown in [MP2] (see also Appendix III) that the space $B.M.O.$, or more generally the spaces defined by these nondecreasing functionals, can be seen as limiting of certain appropriate generalized Trudinger inequalities.

In next result we will use the following notation

$$\|g\|_{L^{r,\infty}(Q)} = \sup_{t>0} t \left( \frac{|\{x \in Q : |g(x)| > t\}|}{|Q|} \right)^{1/r},$$

for the normalized Marcinkiewicz “norm”. Similarly, whenever a weight $w$ is involved we have

$$\|g\|_{L^{r,\infty}(Q,w)} = \sup_{t>0} t \left( \frac{w(\{x \in Q : |g(x)| > t\})}{w(Q)} \right)^{1/r}.$$

**Theorem 2.4.2** Let $w$ be an $A_\infty$ weight and let $a$ be a functional satisfying the (weighted) $D_r$ condition (2.15) for some $r > 0$. Let $f$ be a locally integrable function such that

$$\frac{1}{|Q|} \int_Q |f - f_Q| \leq a(Q),$$  \hspace{1cm} (2.19)

for every cube $Q$. Then

a) There exists a constant $C$ such that for all the cubes $Q$ in $\mathbb{R}^n$

$$\|f - f_Q\|_{L^{r,\infty}(Q,w)} \leq C a(Q).$$  \hspace{1cm} (2.20)

b) Let $p$ with $0 < p < r$. Then, as a consequence of a) there exists a constant $C$ such that for all the cubes $Q$ in $\mathbb{R}^n$ we have

$$\left( \frac{1}{w(Q)} \int_Q |f(y) - f_Q|^p \, w(y) \, dy \right)^{1/p} \leq C a(Q).$$  \hspace{1cm} (2.21)

**Remark 2.4.3** We emphasize that the natural exponent $p = r$ is not attained in the strong inequality (2.21). In fact it is an open question whether the $D_r$ condition is sufficient to assure the strong $L^r$ result. Under certain extra condition the answer is yes as the next corollary shows where a very special selfimproving functional is considered.
Remark 2.4.4 It is not hard to see that we can replace the Lebesgue measure by any doubling measure \( \mu \). However, it is much more interesting the fact that we can go beyond and consider more general measures which are nondoubling. Indeed it has been first shown in [MMNO] how to work with measures that do not “see” hyperplanes parallel to the axes. In particular this is the case of measures \( \mu \) satisfying the growth condition \( \mu(Q) \leq C \ell(Q)^\alpha \) for some positive \( \alpha \). These measures have a growing interest since they arise in the study of the Cauchy transform along curves and its generalizations (see the recent papers [NTV] [Tor]). The \( A_p(\mu) \) theory of weights has been developed in [OP] with applications in the spirit of Theorem 2.4.2 and to singular integrals.

Remark 2.4.5 It is interesting to compare our situation with that in the context of the Campanato-Morrey spaces. These spaces are defined as those functions \( f \) satisfying (2.19) with the functional \( a(Q) = |Q|^{-\delta} \), where \( \delta \) is positive. It is well known that this condition is equivalent to requiring that
\[
\frac{1}{|Q|} \int_Q |f| \leq C |Q|^{-\delta}.
\]
for all cubes \( Q \). This same equivalence holds when the \( L^1 \) norm is replaced by the \( L^p \) norm. This means that the cancellation does not play any role in these spaces. Indeed there are examples by Piccinini showing that these spaces do not have the self-improving properties as derived above.

Corollary 2.4.6 Let \( f \) and \( g \) be locally integrable functions and let \( g \) be an \( A_\infty \) weight. Let \( 0 < \alpha < n \), and suppose that \( f \) satisfies the following inequality for all cubes:
\[
\frac{1}{|Q|} \int_Q |f - f_Q| \leq C \frac{\ell(Q)^\alpha}{|Q|} \int_Q g.
\]
Then with \( r = \frac{n}{n-\alpha} \),
\[
\|f - f_Q\|_{L^{r,\infty}(Q)} \leq C \frac{\ell(Q)^\alpha}{|Q|} \int_Q g.
\]
If furthermore the function \( g \) also satisfies the \( A_\infty \) condition then
\[
\left( \frac{1}{|Q|} \int_Q |f - f_Q|^r \right)^{1/r} \leq C \frac{\ell(Q)^\alpha}{|Q|} \int_Q g.
\]

Proof of Theorem 2.4.2: Part b) follows from Kolmogorov’s inequality: let \( 0 < q < r \) we have that for nonnegative \( g \)
\[
\left( \frac{1}{w(Q)} \int_Q g(x)^q w dx \right)^{1/q} \leq \left( \frac{r}{r - q} \right)^{1/q} \|g\|_{L^{r,\infty}(Q,w)}.
\]
See [GCRdF] p. 485 for instance. We just need to apply part a) to this inequality to \( g = |f - f_Q| \). To prove part a) we fix a cube \( Q \). Observe that we may assume that \( f_Q = 0 \). Hence we have to prove that
\[
\nu^r \frac{w(\{x \in Q : |f(x)| > t\})}{w(Q)} \leq C^r a(Q)^r,
\]

\[
\text{Remark 2.4.4 It is not hard to see that we can replace the Lebesgue measure by any doubling measure } \mu. \text{ However, it is much more interesting the fact that we can go beyond and consider more general measures which are nondoubling. Indeed it has been first shown in [MMNO] how to work with measures that do not “see” hyperplanes parallel to the axes. In particular this is the case of measures } \mu \text{ satisfying the growth condition } \mu(Q) \leq C \ell(Q)^\alpha \text{ for some positive } \alpha. \text{ These measures have a growing interest since they arise in the study of the Cauchy transform along curves and its generalizations (see the recent papers [NTV] [Tor]). The } A_p(\mu) \text{ theory of weights has been developed in [OP] with applications in the spirit of Theorem 2.4.2 and to singular integrals.}

\[
\text{Remark 2.4.5 It is interesting to compare our situation with that in the context of the Campanato-Morrey spaces. These spaces are defined as those functions } f \text{ satisfying (2.19) with the functional } a(Q) = |Q|^{-\delta} \text{, where } \delta \text{ is positive. It is well known that this condition is equivalent to requiring that}
\]
\[
\frac{1}{|Q|} \int_Q |f| \leq C |Q|^{-\delta}.
\]
\text{for all cubes } Q. \text{ This same equivalence holds when the } L^1 \text{ norm is replaced by the } L^p \text{ norm. This means that the cancellation does not play any role in these spaces. Indeed there are examples by Piccinini showing that these spaces do not have the self-improving properties as derived above.}

\[
\text{Corollary 2.4.6 Let } f \text{ and } g \text{ be locally integrable functions and let } g \text{ be an } A_\infty \text{ weight. Let } 0 < \alpha < n, \text{ and suppose that } f \text{ satisfies the following inequality for all cubes:}
\]
\[
\frac{1}{|Q|} \int_Q |f - f_Q| \leq C \frac{\ell(Q)^\alpha}{|Q|} \int_Q g.
\]
\text{Then with } r = \frac{n}{n-\alpha},
\]
\[
\|f - f_Q\|_{L^{r,\infty}(Q)} \leq C \frac{\ell(Q)^\alpha}{|Q|} \int_Q g.
\]
\text{If furthermore the function } g \text{ also satisfies the } A_\infty \text{ condition then}
\]
\[
\left( \frac{1}{|Q|} \int_Q |f - f_Q|^r \right)^{1/r} \leq C \frac{\ell(Q)^\alpha}{|Q|} \int_Q g.
\]

\[
\text{Proof of Theorem 2.4.2: Part b) follows from Kolmogorov’s inequality: let } 0 < q < r \text{ we have that for nonnegative } g
\]
\[
\left( \frac{1}{w(Q)} \int_Q g(x)^q w dx \right)^{1/q} \leq \left( \frac{r}{r - q} \right)^{1/q} \|g\|_{L^{r,\infty}(Q,w)}.
\]
\text{See [GCRdF] p. 485 for instance. We just need to apply part a) to this inequality to } g = |f - f_Q|. \text{ To prove part a) we fix a cube } Q. \text{ Observe that we may assume that } f_Q = 0. \text{ Hence we have to prove that}
\]
\[
\nu^r \frac{w(\{x \in Q : |f(x)| > t\})}{w(Q)} \leq C^r a(Q)^r,
\]
with $C$ independent of $Q$, and $t$.

We will try to get an appropriate good--$\lambda$ inequality (2.28). Good--$\lambda$ inequalities is an important tool discovered by D. L. Burkholder and R. F. Gundy in [BG] (see also [Tor]).

Now, for each $t > 0$, we let $\Omega_t = \{x \in Q : Mf(x) > t\}$ where $M$ will denote in this proof the dyadic Hardy–Littlewood maximal function relative to $Q$ (See Appendix I). Then by the Lebesgue differentiation theorem

$$\{x \in Q : |f(x)| > t\} \subset \Omega_t.$$  

We will assume first that $t > a(Q)$. Hence

$$t > a(Q) \geq \frac{1}{|Q|} \int_Q |f|$$

and we can consider the Calderón–Zygmund covering lemma of $f$ relative to $Q$ for these values of $t$. This yields a collection of dyadic subcubes of $Q$, $\{Q_i\}$, maximal with respect to inclusion, satisfying $\Omega_t = \cup_i Q_i$ and

$$t < \frac{1}{|Q_i|} \int_{Q_i} |f| \leq 2^n t$$

for each $i$.

Now let $q > 1$ a big enough number that will be chosen in a moment. Since $\Omega_{qt} \subset \Omega_t$, we have

$$w(\Omega_{qt}) = w(\Omega_{qt} \cap \Omega_t) = \sum_i w(\{x \in Q_i : Mf(x) > qt\}) = \sum_i w(\{x \in Q_i : M(f\chi_{Q_i})(x) > qt\})$$

where the last equation follows by the maximality of each of the cubes $Q_i$. Indeed, for any of these $i$’s we have

$$Mf(x) = \max\left\{\sup_{P \subseteq Q_i} \frac{1}{|P|} \int_P |f|, \sup_{P \supset Q_i} \frac{1}{|P|} \int_P |f| \right\}$$

$$= \sup_{P \subseteq Q_i} \frac{1}{|P|} \int_P |f| = M(f\chi_{Q_i})(x),$$

since by the maximality of the cubes $Q_i$ when $P$ is dyadic (relative to $Q$) containing $Q_i$ then

$$\frac{1}{|P|} \int_P |f| \leq t.$$

On the other hand for arbitrary $x$,

$$|f(x)| \leq |f(x) - f_{Q_i}| + |f_{Q_i}| \leq |f(x) - f_{Q_i}| + \frac{1}{|Q_i|} \int_{Q_i} |f| \leq |f(x) - f_{Q_i}| + 2^n t$$
and then for \( q > 2^n \)

\[
w(\Omega_{qt}) \leq \sum_i w(E_{Q_i}),
\]

where \( E_{Q_i} = \{ x \in Q_i : M((f - f_{Q_i})\chi_{Q_i})(x) > (q - 2^n)t \} \).

Let \( \epsilon > 0 \) to be chosen in a moment. We split the family \( \{Q_i\} \) in two:

(i) \( i \in I \) if

\[
\frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}| < \epsilon t
\]

or (ii) \( i \in II \) if

\[
\frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}| \geq \epsilon t
\]

Then

\[
w(\Omega_{qt}) \leq \sum_i w(E_{Q_i}) \leq \sum_{i \in I} w(E_{Q_i}) + \sum_{i \in II} w(E_{Q_i}) = I + II.
\]

For I we use that \( M \) is of weak type \((1, 1)\) (with constant one) to control the unweighted part:

\[
|E_{Q_i}| \leq \frac{1}{(q - 2^n)t} \frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}| \leq \frac{\epsilon}{(q - 2^n)} |Q_i|.
\]

Recall that the \( A_\infty \) condition is characterized by

\[
\frac{w(E)}{w(Q)} \leq c \left( \frac{|E|}{|Q|} \right)^\rho,
\]

for some \( c \) and \( \rho \) and independent of \( Q \) and \( E \subset Q \). Therefore we get for each \( i \in I \)

\[
w(E_{Q_i}) \leq C \epsilon^\delta w(Q_i)
\]

for some positive \( \delta \). Then

\[
I \leq C \epsilon^\theta \sum_{i \in I} w(Q_i) \leq C \epsilon^\theta w(\bigcup_i Q_i) = C \epsilon^\theta w(\Omega_t).
\]

For II we use the hypothesis on \( a \) (2.15)

\[
II \leq \sum_{i \in II} w(Q_i) \leq \sum_{i \in II} \left( \frac{1}{t \epsilon |Q_i|} \int_{Q_i} |f - f_{Q_i}| \right)^r w(Q_i)
\]

\[
\leq \frac{1}{t^r \epsilon^r} \sum_{i \in II} a(Q_i)^r w(Q_i) \leq \frac{||a||^r}{t^r \epsilon^r} a(Q)^r w(Q).
\]

Combining all these estimates we have that for \( q > 2^n \), \( t > a(Q) \)

\[
(qt)^r \frac{w(\Omega_{qt})}{w(Q)} \leq \frac{\epsilon^\theta q^r}{(q - 2^n) t^r} \frac{w(\Omega_t)}{w(Q)} + \frac{||a||^r}{\epsilon^r} a(Q)^r.
\]  \hspace{1cm} (2.27)

However, if we choose \( \epsilon \leq ||a|| \) we have the same inequality holds for \( t \leq a(Q) \). In conclusion we have the following inequality:
For all \( q > 2^n, t > 0 \) and \( 0 < \epsilon \leq \|a\| \) we have
\[
(qt)^r \frac{w(\Omega_t)}{w(Q)} \leq \frac{e^n q^r}{(q-2^n)} t^r \frac{w(\Omega_t)}{w(Q)} + \frac{\|a\|^r q^r}{\epsilon^r} a(Q)^r. \tag{2.28}
\]

To conclude we use a standard good–\( \lambda \) method. For \( N > 0 \) we let
\[
\varphi(N) = \sup_{0 < t < N} t^r \frac{w(\Omega_t)}{w(Q)},
\]
which is finite since it is bounded by \( N^r \). Since \( \varphi \) is increasing (2.28) clearly implies
\[
\varphi(N) \leq \varphi(Nq) \leq \frac{e^n q^r}{q - 2^n} \varphi(N) + \frac{\|a\|^r q^r}{\epsilon^r} a(Q)^r.
\]

We now conclude by choosing \( \epsilon \) small enough such that \( \frac{e^n q^r}{q - 2^n} < 1 \), and letting \( N \to \infty \).

We conclude this section by pointing out that as a corollary of the proof of above theorem we can deduce a key estimate relating a function and the action of the sharp maximal operator \( M^\# \) on it. This is perhaps the best way of showing the relationship between a function and its smoothness. Recall that the sharp maximal operator \( M^\# \) is an operator introduced by C. Fefferman and E. Stein and it is defined as follows:

\[
M^\# f(x) = \sup_{x \in Q} \inf_c \frac{1}{|Q|} \int_Q |f(y) - c| \, dy
\]

which is equivalent to
\[
\sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy \approx \sup_{x \in Q} \frac{1}{|Q|^2} \int_Q \int_Q |f(y) - f(x)| \, dy \, dx
\]
where as usual the supremum is taken over all the cubes containing \( x \).

In order to derive a global result we need to consider spaces on which we can make the global Calderón–Zygmund covering lemma (cf. Appendix I):
\[
CZ = \{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : f_Q \to 0 \text{ when } Q \uparrow \mathbb{R}^n \}
\]
It is easy to see that \( \cup_{p \geq 1} L^p \subset CZ \).

**Theorem 2.4.7** Let \( 0 < p < \infty \) and let \( w \) be a \( A_\infty \) weight. Then

a) Suppose that \( f \) be an integrable function on a cube \( Q \) of \( \mathbb{R}^n \). Then there exists a constant \( C \) independent of \( f \) such that
\[
\int_Q |f - f_Q|^p w \, dx \leq C [w]_{A_\infty} \int_Q (M^\# f)^p w \, dx \tag{2.29}
\]

b) Suppose that \( f \in CZ \). Then there exists a constant \( C \) independent of \( f \) such that
\[
\int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \leq C [w]_{A_\infty} \int_{\mathbb{R}^n} (M^\# f(x))^p w(x) \, dx. \tag{2.30}
\]
2.5 Proof of the Poincaré-Sobolev inequalities

The purpose of this section is to prove the Poincaré-Sobolev inequalities that are in presented Section 2.2. The first part of next result is classical (see [EG]). However, our point of view is different and can be extended to other situations such as the case of subelliptic operators [FPW1] [MP1]. The second part is not that well known.

Recall that \( p^* = \frac{pn}{n-p} \) denotes the Sobolev exponent of \( p \).
Theorem 2.5.1 Let \( 1 \leq p < n \).

a) Let \( f \) be Lipschitz function, then
\[
\left( \frac{1}{|Q|} \int_Q |f - f_Q|^p \right)^{1/p} \leq C \ell(Q) \left( \frac{1}{|Q|} \int_Q |\nabla f|^p \right)^{1/p}.
\] (2.31)

b) Let \( w \in A_p \), and let \( f \) be a Lipschitz function, then there exists a positive constant \( \delta \) depending upon the \( A_p \) constant of \( w \) such that
\[
\left( \frac{1}{w(Q)} \int_Q |f - f_Q|^{p(n' + \delta)} w \right)^{1/p(n' + \delta)} \leq C \ell(Q) \left( \frac{1}{w(Q)} \int_Q |\nabla f|^p w \right)^{1/p}.
\] (2.32)

Proof/Discussion: a) Define the following functional
\[ a(Q, f) = \ell(Q) \left( \frac{1}{|Q|} \int_Q |\nabla f|^p \right)^{1/p} , \]
as a function of both \( Q \) and \( f \). Observe that \( a \) satisfies the (unweighted) \( D_p \) condition (see the calculation 2.16 from previous section) with constant equal one, in particular is uniform on \( f \). Hence, we may apply Theorem 2.4.2 and get the weak type estimate
\[
\| f - f_Q \|_{L^{p^*}, \infty(Q, w)} \leq C a(Q, f),
\]
with constant \( C \) independent of \( Q \) and \( f \). What we need to do is to pass from this weak type inequality to the corresponding strong type estimate. To accomplish this, we use some ideas taken from R. Long and F. Nie in [LN]. We remit [FPW1] for details and generalizations\(^1\). The key point is to have a gradient type functional on the right hand side. We will illustrate this in the proof of a (global) sharp two weight isoperimetric inequality Theorem 5.1.1 from Chapter 5.

b) We start with the \((1, 1)\) Poincaré inequality
\[
\frac{1}{|Q|} \int_Q |f - f_Q| \leq C \frac{\ell(Q)}{|Q|} \int_Q |\nabla f|.
\]
By the \( A_p \) condition using part b) of Theorem 1.0.6 we have that
\[
\frac{1}{|Q|} \int_Q |f - f_Q| \leq C \ell(Q) \left( \frac{1}{w(Q)} \int_Q |\nabla f|^p w \right)^{1/p} = a(Q, f).
\]

\(^1\)P. Koskela informed me that Long-Nie method is somehow hidden in [Maz], although not in the way presented here; also it has been rediscovered by some other authors [BCLSC]
Also by part b) of Theorem 1.0.6 we have that
\[
\left( \frac{|P|}{|Q|} \right)^p \leq C \frac{w(P)}{w(Q)}
\]
for any subcube \( P \subset Q \) or what is the same
\[
\frac{\ell(P)}{\ell(Q)} \leq C \left( \frac{w(P)}{w(Q)} \right)^{1/np}.
\]

By the openness property of the \( A_p \) condition (8.7) there is small \( \epsilon \) such that \( w \in A_{p-\epsilon} \). Then \( w \) satisfies
\[
\frac{\ell(P)}{\ell(Q)} \leq c \left( \frac{w(P)}{w(Q)} \right)^{1/(np-\epsilon)}.
\]

Now let \( q = \lambda p \) with \( \lambda > 1 \) such that
\[
\frac{1}{n(p-\epsilon)} = \frac{1}{p} - \frac{1}{q} = \frac{1}{p\lambda'}
\]
with \( \lambda = \left( \frac{p-\epsilon}{p} \right)^{\prime} = \lambda = n' + \delta \) for some \( \delta > 0 \). Then we have that
\[
\frac{\ell(P)}{\ell(Q)} \left( \frac{w(P)}{w(Q)} \right)^{1/q} \leq c \left( \frac{w(P)}{w(Q)} \right)^{1/p}.
\]

Using this property it is easy to check that \( a(Q, f) \) satisfies the \( D_q \) condition with a constant independent of \( Q \) and \( f \). As in part a) we can pass from the weak type estimate which follows from Theorem 2.4.2 to obtain the desired result
\[
\left( \frac{1}{w(Q)} \int_Q |f-f_Q|^q w \right)^{1/q} \leq C a(Q, f).
\]
\[
(2.33)
\]

\[\square\]

Mejorar este trozo que se incorpora despues:

These results are not new, we know give applications to derive new results. In particular we improve this theorem at the endpoint \( p = 1 \) in two ways. We also improve the result given in p. 308 [HKM] since we are able to reach both endpoints: \( p = 1 \) and for the Sobolev exponent by \( p^* = \frac{pn}{n-p} \), \( 1 \leq p < n \). The result is even new in the euclidean setting.

**Theorem 2.5.2** Also let \( w \) be a weight function such that \( w \in A_1 \). Then, we have
\[
\left( \frac{1}{w(Q)} \int_Q |f-f_Q|^{p^*} w \right)^{1/p^*} \leq C \ell(Q) \left( \frac{1}{w(Q)} \int_Q |\nabla f|^p w \right)^{1/p}
\]
\[
(2.34)
\]
with \( C \) independent of \( f, Q \).
Observe that this result sharpen the previous theorem when $p = 1$.

**Proof:** We follow here [LP2] where the result is given in a more general setting and for higher order derivatives.

Let $w \in A_1$ and we begin again by using the (1,1) Poincaré inequality

$$\frac{1}{|Q|} \int_Q |f - f_Q| \leq C \ell(Q) \frac{1}{|Q|} \int_Q |\nabla f|$$

Now, since $A_1 \subset A_p$ we have that the right hand side is less or equal than

$$\ell(Q) \left( \frac{1}{w(Q)} \int_Q |\nabla f|^p w \right)^{1/p} = b(Q, f) = b(Q)$$

It will enough to check the $D_p^*$ condition and for which it is enough to check

$$\frac{\ell(Q_1)}{\ell(Q_2)} \left( \frac{w(Q_1)}{w(Q_2)} \right)^{1/p^*} \leq C \left( \frac{w(Q_1)}{w(Q_2)} \right)^{1/p}$$

for all cubes $Q_1, Q_2$ such that $Q_1 \subset Q_2$. But this is equivalent to

$$\left( \frac{\ell(Q_1)}{\ell(Q_2)} \right)^n \leq C \frac{w(Q_1)}{w(Q_2)}$$

since $\frac{1}{n} = \frac{1}{p} - \frac{1}{p^*}$. Now by the doubling property it will be enough to check

$$\frac{|Q_1|}{|Q_2|} \leq C \frac{w(Q_1)}{w(Q_2)}$$

and this follows from the fact that $w \in A_1$. 

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Chapter 3
Rearrangements

The main purpose of this chapter to improve the main result in the previous chapter by means of a new technique based on rearrangements. This method yields not only yields stronger but also provide new direct proofs avoiding the use the classical good-\(\lambda\) inequality of Burkholder and Gundy as in described in the previous chapter.

The natural question that we consider is the following: Is it possible to relax the \(L^1\) norm in \((2.19)\) to derive the same self-improving property? In this Chapter, we study this issue and weaken the hypothesis to obtain such self-improving property. More precisely, we will show that the \(L^1\) norm can be replaced by any \(L^q\) quasi-norm with \(0 < q < 1\) (See Corollaries 3.0.4 and 3.0.5 below). Even further, we will show that the left hand side of \((2.19)\) can be replaced by a weaker expression which is defined in terms of non-increasing rearrangements (see below for the precise definition).

Our main result is the following.

**Theorem 3.0.3** Let \(\omega\) be an \(A_\infty\) weight and let \(a\) be a functional satisfying the \(D_r(\omega)\) condition \((2.15)\) for some \(r > 0\). Suppose that \(f\) is a measurable function such that for any cube \(Q\)

\[
\inf_{\alpha} \left( (f - \alpha) \chi_Q \right)^* (\lambda |Q|) \leq c_a a(Q), \quad 0 < \lambda < 1. \tag{3.1}
\]

Then the same consequences of Theorem 2.4.2 holds namely, there exists a constant \(c\) such that for any cube \(Q\)

\[
\|f - f_Q\|_{L^r,\infty(Q,\omega)} \leq c a(Q). \tag{3.2}
\]

Recall that the non-increasing rearrangement \(f^*\) of a measurable function \(f\) is defined by

\[
f^*(t) = \inf \{ \lambda > 0 : \mu_f(\lambda) < t \}, \quad t > 0,
\]

where

\[
\mu_f(\lambda) = |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|, \quad \lambda > 0,
\]

is the distribution function of \(f\).
An important fact is that
\[ \int_{\mathbb{R}^n} |f|^p \, dx = \int_0^\infty f^*(t)^p \, dt, \]
or more generally if \( E \) is any measurable function:
\[ \int_E |f|^p \, dx = \int_0^{|E|} f^*(t)^p \, dt, \]

If \( f \) is only a measurable function and if \( Q \) is a cube then we define the following quantity:
\[ (f\chi_Q)^*(\lambda|Q|), \quad 0 < \lambda \leq 1. \]

We can think of this expression as another way of averaging the function. Indeed for any \( p > 0 \), and \( 0 < \lambda \leq 1 \)
\[ (f\chi_Q)^*(\lambda|Q|) \leq \left( \frac{1}{\lambda|Q|} \int_Q |f|^p \, dx \right)^{1/p}. \]
Hence we have the following.

**Corollary 3.0.4** Let \( f, a \) and \( \omega \) as in Theorem 3.0.3. Suppose that for some \( 0 < p < 1 \) and for any cube \( Q \)
\[ \inf_{\alpha} \left( \frac{1}{|Q|} \int_Q |f - \alpha|^p \, dx \right)^{1/p} \leq a(Q). \]
Then there is a geometric constant \( c \) such that for any cube \( Q \)
\[ \|f - f_Q\|_{L^{\infty}(Q,\omega)} \leq c a(Q). \]

Using that for any measurable function \( f \), and any \( 0 < \delta < 1 \)
\[ \left( \frac{1}{|Q|} \int_Q |f|^p \, dx \right)^{1/p} \leq c_p \|f\|_{L^{1,\infty}(Q)}. \]

**Corollary 3.0.5** If as above we suppose that for any cube \( Q \)
\[ \|f - f_Q\|_{L^{1,\infty}(Q)} \leq a(Q), \]
then there is a geometric constant \( c \) such that for any cube \( Q \)
\[ \|f - f_Q\|_{L^{r,\infty}(Q,\omega)} \leq c a(Q). \]
Our proof of Theorem 3.0.3 is essentially based on the following two ingredients. The first one is a relation between rearrangements and oscillations of a function $f$. This technique goes back to [BDVS], and it was further developed in [L1, L2]. We also use some ideas from the works [J, S]. The second ingredient is an appropriate covering lemma of Calderón-Zygmund type. In the context of the standard Euclidean space $\mathbb{R}^n$ with doubling measure such covering lemmas can be obtained simply by application of the usual Calderón-Zygmund lemma to characteristic functions (see, e.g., [BDVS, L1]. In the case of $\mathbb{R}^n$ with non-doubling measure a covering lemma of such type has been recently proved in [MMNO]. Our covering lemma, in the context of spaces of homogeneous type, is presented in Section 3 below.

The main result of this paper can be applied to improve the results of [MP2, OP]. In [MP2], an exponential self-improving property was established assuming (2.19) with a functional $a$ satisfying the so-called $T_p$ condition, stronger than the $D_r$ condition. In Chapter 9, we will improve this result by relaxing the initial assumption (2.19) as in Theorem 3.0.3. In [OP], a non-homogeneous variant of Theorem 2.4.2 was proved in the context of $\mathbb{R}^n$ with any (non-doubling) measure. We will show that this result can be also proved under similar relaxed assumptions.

We also obtain an analogue of the main results of [L1, L2] concerning the so-called local sharp maximal function, in the context of spaces of homogeneous type.

### 3.1 Rearrangements

We mention here some simple properties of rearrangements. It follows easily from the definition that

$$\mu\{x \in \mathbb{R}^n : |f(x)| > f^*_\mu(t)\} \leq t \quad \text{and} \quad \mu\{x \in \mathbb{R}^n : |f(x)| \geq f^*_\mu(t)\} \geq t.$$

Next, for any $0 < \lambda < 1$,

$$(f + g)_\mu^*(t) \leq f^*_\mu(\lambda t) + g^*_\mu((1 - \lambda)t). \quad (3.3)$$

For any measurable set $E$ with finite positive measure $\mu$ we have,

$$\inf_E |f| \leq (f\chi_E)_\mu^*(\mu(E)). \quad (3.4)$$

Finally we recall that the (weighted normalized) weak $L^r$ norm is defined by

$$\|g\|_{L^r,\infty(E,\omega)} = \sup_{\lambda > 0} \lambda \left(\frac{\omega(\{x \in E : |g(x)| > \lambda\})}{\omega(E)}\right)^{1/r}$$

or, equivalently,

$$\|g\|_{L^r,\infty(E,\omega)} = \sup_{0 < t \leq \omega(E)} \left(\frac{t}{\omega(E)}\right)^{1/r} (g\chi_E)_\omega^*(t). \quad (3.5)$$

We will need the following two propositions about local rearrangements.
Proposition 3.1.1 For any measurable function $f$, any weight $\omega$, and each measurable set $E \subset \mathbb{R}^n$ with $0 < \omega(E) < \infty$, and for any $0 < \lambda < 1$:

$$(f \chi_E)^\ast(\lambda \omega(E)) \leq 2 \inf_c \left( (f - c) \chi_E)^\ast(\lambda \omega(E)) + (f \chi_E)^\ast((1 - \lambda)\omega(E))\right).$$

Proof:
Essentially the same was proved in [L1] in a less general context. The proof of this proposition follows the same lines, and we shall recall it only for the sake of completeness.

Using (3.3) and (3.4), for any constant $\alpha$ we get

$$|\alpha| \leq \inf_{x \in E} \left( |f(x) - \alpha| + |f(x)| \right) \leq ((f - \alpha) \chi_E)^\ast(\lambda \omega(E)),$$

$$\leq ((f - \alpha) \chi_E)^\ast((1 - \lambda)\omega(E)).$$

From this and from the estimate

$$(f \chi_E)^\ast(\lambda \omega(E)) \leq ((f - \alpha) \chi_E)^\ast(\lambda \omega(E)) + |\alpha|,$$

we get immediately the required inequality.

Proposition 3.1.2 Let $\omega \in A_\infty$. For any cube $Q$ and any measurable function $f$, and for any $0 < \lambda < 1$:

$$(f \chi_Q)^\ast(\lambda \omega(Q)) \leq (f \chi_Q)^\ast\left(\left(\frac{\lambda}{c}\right)^{1/\varepsilon}|Q|\right),$$

where $c$ and $\varepsilon$ are $A_\infty$-constants of $\omega$.

Proof: It follows from the definitions of $A_\infty$ and of the rearrangement that for any $\delta > 0$,

$$\omega\left\{ x \in Q : |f(x)| > (f \chi_Q)^\ast\left(\left(\frac{\lambda}{c}\right)^{1/\varepsilon}|Q|\right) + \delta \right\}$$

$$\leq c \left( \frac{\mu\left\{ x \in Q : |f(x)| > (f \chi_Q)^\ast\left(\left(\frac{\lambda}{c}\right)^{1/\varepsilon}|Q|\right) + \delta \right\}}{|Q|} \right)^\varepsilon \omega(Q) < \lambda \omega(Q),$$

which is equivalent to that

$$(f \chi_Q)^\ast(\lambda \omega(Q)) \leq (f \chi_Q)^\ast\left(\left(\frac{\lambda}{c}\right)^{1/\varepsilon}|Q|\right) + \delta.$$ 

Letting $\delta \to 0$ yields the required inequality.

A relevant class of weights is given by the $A_\infty$ class of Muckenhoupt: if there are positive constants $c, \varepsilon$ such that

$$\omega(E) \leq c \left( \frac{|E|}{|Q|} \right)^\varepsilon \omega(Q) \quad (3.5)$$

for every cube $Q$ and every measurable set $E \subset Q$. 

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3.2 A basic covering lemma

In this section we prove a covering lemma of Calderón-Zygmund which is nothing but a special case of the usual Calderón-Zygmund classical lemma for a cube.

We assume that $\omega$ is dyadic doubling:

$$\omega(Q') \leq d \omega(Q)$$

where $Q'$ is the ancestor of $Q$.

**Lemma 3.2.1** Let $\omega$ be a doubling weight with doubling dyadic constant $d \geq 1$. Then for any $0 < \lambda < 1$, any cube $Q$ and any set $E \subset Q$ with $\omega(E) \leq \lambda \omega(Q)$, there is a countable family of pairwise disjoint dyadic cubes $\{Q_i\}_i$ from $Q$ with $E \subset \bigcup_i Q_i$, except for a $\omega$-null set such that

(i) $\omega(Q_i \cap E) \leq d \lambda \omega(Q_i)$;

(ii) $\omega(Q_i \cap E) \geq \lambda \omega(Q_i)$;

(iii) $\omega(E) \leq d \lambda \sum_i \omega(Q_i)$.

**Proof:** Items (i) and (ii) follow from the weighted version of the Calderón-Zygmund for cubes (see 7.0.5): If

$$\frac{1}{\omega(Q)} \int_Q |f|w \, dx < \lambda$$

there exists mutually disjoint dyadic cubes $\{Q_i\}_i$ (dyadic with respect to $Q$) contained in $Q$ such that

$$\lambda \frac{1}{\omega(Q)} \int_Q |f|w \, dx \leq d \lambda$$

and $|f(x)| \leq \lambda$ for almost $x \in Q \setminus \bigcup_i Q_i$.

Letting $f = \chi_E$ we get (i) and (ii). Observe that since $0 < \lambda < 1$, $w(Q \setminus \bigcup_i Q_i) = 0$ and hence $E \subset \bigcup_i Q_i$, except for a $\omega$-null set. Part (iii) is immediate. □

3.3 Proof of the main Theorem

Adapting ideas from [L1, L2] we prove in this section the main theorem of this Chapter

**Theorem 3.0.3.**

**Proof:**

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The goal is to prove the following estimate: there are constants $c', \lambda$ with $0 < \lambda < c' < \infty$ such that for any cube $Q$ and for any $0 < t < \omega(Q)$

$$
(f \chi_Q)^* (t) \leq c' a(Q) \left( \frac{\omega(Q)}{t} \right)^{\frac{1}{r}} + (f \chi_Q)^* (|Q|).
$$

(3.6)

First we will prove this estimate for small values of $t$.

The key result in the proof is the following estimate for the rearrangement of $f$: there exists a constant $\lambda'$, $0 < \lambda' < 1$ such that for any $0 < t < \lambda' \omega(Q)$

$$(f \chi_Q)^* (t) \leq a(Q) \left( \frac{\omega(Q)}{t} \right)^{\frac{1}{r}} + (f \chi_Q)^* (2t).$$

(3.7)

Throughout the proof $d$ is doubling constant and $a$ and $\varepsilon$ are the constants from the $A_\infty$-condition (3.5) of $\omega$.

For any $t$ we consider the set $E = E_t$

$$E = \{ x \in Q : |f(x)| \geq (f \chi_Q)^* (t) \}.$$  

We can assume that $(f \chi_Q)^* (t) > (f \chi_Q)^* (2t)$, since otherwise (3.7) is trivial. Observe that $t \leq \omega(E) \leq 2t$.

Set $\lambda' = \frac{1}{4d}$ and $\lambda = (\frac{\lambda'}{\varepsilon})^{1/\varepsilon}$ and let $0 < t < \frac{\lambda'}{2} \omega(Q)$. Then we have that

$$\omega(E) \leq \lambda \omega(Q),$$

and we can apply Lemma 3.2.1 to the set $E$ and number $\lambda'$. Hence, we get a countable family of pairwise disjoint balls $\{Q_i\}$ satisfying (i)-(iii) of Lemma 3.2.1. Thus combining item (ii) and Proposition 3.1.1 we have,

$$(f \chi_Q)^* (t) \leq \inf_{x \in E} |f(x)| \leq \inf_i \inf_{x \in E \cap Q_i} |f(x)|$$

$$\leq \inf_i (f \chi_{E \cap Q_i})^* (\omega(E \cap Q_i))$$

$$\leq \inf_i (f \chi_{Q_i})^* (\lambda \omega(Q_i))$$

$$\leq \inf_i \left[ 2 \inf \left( (f - \alpha) \chi_{Q_i} \right)^* (\lambda \omega(Q_i)) \right]$$

$$+ \left( f \chi_{Q_i} \right)^* ((1 - \lambda') \omega(Q_i)).$$

Applying item (i) together with Proposition 3.1.2 we obtain for each $i$

$$\inf_{\alpha} \left( (f - \alpha) \chi_{Q_i} \right)^* (\lambda \omega(Q_i)) \leq \inf_{\alpha} \left( (f - \alpha) \chi_{Q_i} \right)^* (|Q_i|)$$

$$\leq c \lambda a(Q_i),$$

where in the last step we have used the basic initial hypothesis (3.1) to the ball $Q_i$. Hence,

$$(f \chi_Q)^* (t) \leq \inf_i \left( 2c \lambda a(Q_i) + (f \chi_{Q_i})^* ((1 - \lambda') \omega(Q_i)) \right).$$

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Since the balls $Q_i$ are dyadic, disjoint and contained in $Q$ we have by the $D_r$ condition (2.15)

$$\sum_i a(Q_i)^r \omega(Q_i) \leq \|a\|^{r_a} a(Q)^r \omega(Q).$$

Let $M$ a big enough number to be chosen soon. We split the family of cubes as follows:

$i \in I$ if

$$a(Q_i) \leq M^{1/r} \|a\| a(Q) \left( \frac{\omega(Q)}{t} \right)^{\frac{1}{r}}$$

and $i \in II$ if it satisfies the opposite inequality. The proof below shows that the family $I$ is not empty. We claim that

$$\sum_{i \in I} \omega(Q_i) \geq (1 - \lambda' \frac{1}{d\lambda'}) t > 2t,$$

where we choose $M > d\lambda'$. Indeed, by the $D_r$ condition

$$\sum_{i \in II} \omega(Q_i) \leq \frac{t}{M}$$

and hence (3.9) follows from the item (iii) of Lemma 3.2.1 and our choice of $\lambda'$.

Therefore we have

$$(f \chi_{Q_i})^\ast(t) \leq 2c_M M^{1/r} \|a\| a(Q) \left( \frac{\omega(Q)}{t} \right)^{\frac{1}{r}} \inf_{i \in I} \left( f \chi_{Q_i} \right)^\ast \left( (1 - \lambda') \omega(Q_i) \right).$$

Set now for $i \in I$

$$E_i = \{ x \in Q_i : |f(x)| \geq (f \chi_{Q_i})^\ast \left( (1 - \lambda') \omega(Q_i) \right) \}.$$

By (3.9),

$$\omega(\cup_{i \in I} E_i) = \sum_{i \in I} \omega(E_i) \geq (1 - \lambda') \sum_{i \in I} \omega(Q_i) \geq (1 - \lambda')(\frac{1}{d\lambda'} - \frac{1}{M})t > 2t,$$

if we choose for instance $M > 2d\lambda'$ and $0 < \lambda' < \frac{1}{1+4d}$.

Thus,

$$\inf_{i \in I} \left( f \chi_{Q_i} \right)^\ast \left( (1 - \lambda') \omega(Q_i) \right) \leq \inf_{i \in I} \inf_{x \in E_i} |f(x)| = \inf_{x \in \cup_{i \in I} E_i} |f(x)| \leq (f \chi_{Q})^\ast(2t),$$

and hence (3.7) is proved for the values $0 < t < \frac{\lambda'}{2} \omega(Q)$.

Now, for one of these values of $t$, there is $k = 1, \cdots$, such that

$$\frac{\lambda'}{2k+1} \omega(Q) \leq t < \frac{\lambda'}{2k} \omega(Q),$$
and iterating (3.7) yields

\[(f\chi_Q)_\omega^*(t) \leq c\alpha(Q) \left(\frac{\omega(Q)}{t}\right)^{\frac{1}{r}} \sum_{j=0}^{k} (2^{-j})^{\frac{1}{r}} + (f\chi_Q)_\omega^*(2^{k+1}\omega(Q))\]
\[\leq c' a(Q) \left(\frac{\omega(Q)}{t}\right)^{\frac{1}{r}} + (f\chi_Q)_\omega^*(\lambda'\omega(Q)).\]  

(3.10)

Next, we observe that (3.10) trivially holds for \(t \geq \lambda'\omega(Q)\), and hence this formula holds for any \(0 < t < \omega(Q)\).

Applying Proposition 3.1.2 to the second term on the right-hand side of (3.10), we obtain (3.6).

To finish the proof we observe that if \(\alpha\) is any number, \(f - \alpha\) also satisfies the initial assumption (3.1) and hence (3.6) holds for \(f - \alpha\) as well:

\[((f - \alpha)\chi_Q)_\omega^*(t) \leq c' a(Q) \left(\frac{\omega(Q)}{t}\right)^{\frac{1}{r}} + ((f - \alpha)\chi_Q)_\omega^*(\lambda'|Q|).\]

Recalling that

\[\|g\|_{L^{r,\infty}(Q,\omega)} = \sup_{0 < t \leq \omega(Q)} (g\chi_Q)_\omega^*(t) \left(\frac{t}{\omega(Q)}\right)^{1/r},\]

we have by multiplying \(\left(\frac{t}{\omega(Q)}\right)^{\frac{1}{r}}\) and taking the supremum over \(0 < t < \omega(Q)\) that

\[\|f - \alpha\|_{L^{r,\infty}(Q,\omega)} \leq c' a(Q) + ((f - \alpha)\chi_Q)_\omega^*(\lambda'|Q|).\]

Taking the infimum over all \(\alpha\) and using again (3.1) combined with the \(D_r\) condition we have

\[\inf_{\alpha} \|f - \alpha\|_{L^{r,\infty}(Q,\omega)} \leq c' a(Q) + \inf_{\alpha} ((f - \alpha)\chi_Q)_\omega^*(\lambda'|Q|)\]
\[\leq c' a(Q) + c_\lambda a(Q)\]
\[\leq c'' a(Q).\]

The proof is now complete. 

\[\square\]
Chapter 4

Orlicz maximal functions

4.1 Orlicz spaces

To derive some sharp weighted estimates in Chapters (5) and (6) we need to consider certain maximal functions defined in terms of Orlicz spaces. In this section we recall some basic definitions and facts about Orlicz spaces, referring to [RR] and [KJF] for a complete account.

A function $B : [0, \infty) \rightarrow [0, \infty)$ is called a Young function if it is continuous, convex, increasing and satisfies $B(0) = 0$ and $B(t) \rightarrow \infty$ as $t \rightarrow \infty$. It follows that $B(t)/t$ is increasing. Some relevant examples are $B(t) = t^p (\log(1 + t))^\alpha$, $1 \leq p < \infty$, $\alpha > 0$ or $B(t) = \Phi(t) = \exp t^p - 1$, $1 \leq p < \infty$.

For Orlicz norms we are usually only concerned about the behavior of Young functions for $t$ large. Given two functions $B$ and $C$, we write $B(t) \approx C(t)$ if $B(t)/C(t)$ is bounded and bounded below for $t \geq c > 0$.

By definition, the Orlicz space $L_B$ consists of all measurable functions $f$ such that

$$\int_{\mathbb{R}^n} B\left(\frac{|f|}{\lambda}\right) d\mu < \infty$$

for some positive $\lambda$. The space $L_B$ is a Banach function space with the Luxemburg norm defined by

$$\|f\|_B = \inf\{\lambda > 0 : \int_{\mathbb{R}^n} B\left(\frac{|f|}{\lambda}\right) d\mu \leq 1\}.$$

Each Young function $B$ has an associated complementary Young function $\bar{B}$ satisfying

$$t \leq B^{-1}(t)\bar{B}^{-1}(t) \leq 2t$$

for all $t > 0$. The function $\bar{B}$ is called the conjugate of $B$, and the space $L_{\bar{B}}$ is called the conjugate space of $L_B$. For example, if $B(t) = t^p$ for $1 < p < \infty$, then $\bar{B}(t) = t^{p'}$, $p' = p/(p - 1)$, and the conjugate space of $L^p(\mu)$ is $L^{p'}(\mu)$. Another example

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that will be used frequently is $B(t) \approx t^p (\log t)^{-1-\epsilon}$, $1 < p < \infty$, $\epsilon > 0$ with complementary function $\bar{B}(t) \approx t^p (\log t)^{(p'-1)(1+\epsilon)}$ (cf. [O]).

A very important property of Orlicz (or, more generally, Banach function spaces) is the generalized Hölder inequality

$$\int_{\mathbb{R}^n} |fg| d\mu \leq \|f\|_B \|g\|_{\bar{B}}.$$  \hspace{1cm} (4.2)

We will always assume that the Young function $B$ satisfies the doubling condition $B(2t) \leq C B(t)$. If $B$ is doubling then $B'(t) \approx B(t)/t$ almost everywhere.

$L_B$ is a rearrangement–invariant space with fundamental function (see [BS]) given by

$$\varphi_B(t) = \varphi_{B|L^p(\mu)}(t) = \frac{1}{B^{-1}(\frac{1}{t})}.$$ \hspace{1cm} (4.3)

In particular if $E$ is a measurable subset of $X$, then

$$\|\chi_E\|_{B,\mu} = \frac{1}{B^{-1}(\frac{1}{\mu(E)})}.$$ \hspace{1cm} (4.4)

### 4.2 Orlicz maximal functions

We begin the section by defining a class of Young functions that will play a key role in Chapters 5 and 6. We need to define an appropriate maximal in terms of Orlicz norm. Given a Young function $B$ and a cube $Q$, the mean Luxembourg norm of a measurable function $f$ on $Q$ is defined by

$$\|f\|_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B\left(\frac{|f|}{\lambda}\right) dx \leq 1 \right\}.$$ \hspace{1cm} (4.5)

When $B(t) = t \log(1 + t)^{p-1+\delta}$, this norm is also denoted by $\| \cdot \|_{L(\log L)^{p-1+\delta},Q}$.

Given a Young function $B$, define the maximal function

$$M_B f(x) = \sup_{Q \ni x} \|f\|_{B,Q}.$$ 

**Definition 4.2.1** Let $1 \leq p < \infty$. A nonnegative function $B(t), t > 0$, satisfies the $B_p$ condition if there is a constant $c > 0$ such that

$$\int_c^\infty \frac{B(t)}{t^p} \frac{dt}{t} \approx \int_c^\infty \left( \frac{t^p}{B(t)} \right)^{p-1} \frac{dt}{t} < \infty.$$ 

Simple examples of functions which satisfy $B_p$ are $t^{p-\delta}$ and $t^p (\log(1 + t))^{-1-\delta}$, both when $\delta > 0$.

The interest of this class of Young functions can be seen from the following lemma taken from [P1]. It is used in this paper to derive sharp two–weight norm inequalities for the Hardy-Littlewood maximal function.
Lemma 4.2.2  Let $1 < p < \infty$. Then the following statements are equivalent.

a) $B$ is a Young doubling function satisfying the $B_p$ condition;
b) there is a constant $c$ such that
\[
\int_{\mathbb{R}^n} M_B f(y)^p \, dy \leq c \int_{\mathbb{R}^n} f(y)^p \, dy \tag{4.6}
\]
for all nonnegative, locally integrable functions $f$;
c) there is a constant $c$ such that
\[
\int_{\mathbb{R}^n} Mf(y)^p \, dy \leq c \int_{\mathbb{R}^n} \frac{1}{[Mf(u)(y)]^p} \, dy \leq c \int_{\mathbb{R}^n} \frac{f(y)^p}{u(y)^p} \, dy, \tag{4.7}
\]
for all nonnegative functions $f$ and $u$.

Proof: To prove b) we will use the classical approach (cf. for instance [GCRdF] Ch. 2.). Observe that we may assume that $f \geq 0$. For $t > 0$ we let $\Omega_t = \{x \in \mathbb{R}^n : M_B f(x) > t\}$. We claim that
\[
|\Omega_t| \leq C \int_{\mathbb{R}^n} B \left( \frac{f(y)}{t} \right) \, dy \tag{4.8}
\]
Assume the claim for the moment. Using (4.8) together with the fact that $M_B$ is bounded on $L^\infty$ we write $f$ as $f = f_1 + f_2$, where $f_1(x) = f(x)$ if $f(x) > \frac{t}{2}$, and $f_1(x) = 0$ otherwise. Then $M_B f(x) \leq M_B f_1(x) + M_B f_2(x) \leq M_B f_1(x) + \frac{t}{2}$, and it follows immediately that
\[
|\Omega_t| \leq C \int_{x: f(x) > t/2} B \left( \frac{f(x)}{t} \right) \, dx
\]
Hence, combining this with the change of variable $t = \frac{f(y)}{x}$ yield
\[
\int_{\mathbb{R}^n} M_B f(y)^p \, dy = p \int_0^\infty t^p |\{ y \in \mathbb{R}^n : M_B f(y) > t\}| \frac{dt}{t} \leq
\]
\[
\leq C \int_0^\infty t^p \int_{\{ y \in \mathbb{R}^n : f(y) > t/2\}} B \left( \frac{f(y)}{t} \right) \, dy \, dt = C \int_{\mathbb{R}^n} \int_0^{2f(y)} t^p B \left( \frac{f(y)}{t} \right) \, dt \, dy =
\]
\[
= C \int_{\mathbb{R}^n} f(y)^p \, dy \int_1^{\infty} \frac{B(t)}{t^{p-1}} \, dt = C \int_{\mathbb{R}^n} f(y)^p \, dy,
\]
since $B \in B_p$.
The proof of the claim follows easily from the Vitali covering lemma 1.0.4. Indeed, assuming that $\Omega_t$ is not empty let $K$ be any compact contained in $\Omega_t$. Let $x \in K$, then by definition of the maximal function there is a cube $Q = Q_x$ containing $x$ such that
\[ \|f\|_{B,Q} > t, \]
or what is the same
\[ \frac{1}{|Q|} \int_Q B\left(\frac{f}{t}\right) > 1. \]
Then $K \subset \bigcup_{x \in K} Q_x$ and by compactness we can extract a finite family of cubes $\mathcal{F} = \{Q\}$ such that $K \subset \bigcup_{Q \in \mathcal{F}} Q$ and where each cube satisfies (4.9). Then by Vitali’s lemma we can extract a pairwise disjoint cubes $\{Q_j\}_{j=1}^M$ such that $K \subset \bigcup_{Q \in \mathcal{F}} Q$ and where each cube satisfies (4.9). Then by definition of $\|f\|_{B,Q}$
\[ |K| \leq C \sum_{j=1}^M |Q_j| \leq C \sum_{j=1}^M \int_{Q_j} B\left(\frac{f}{t}\right) \leq C \int_{\mathbb{R}^n} B\left(\frac{f}{t}\right). \]
This proves that a) implies b).

Let us assume that c) holds. Observe that (4.7) is equivalent to
\[ \int_{\mathbb{R}^n} M(fg)(y)^p \frac{1}{[M_B(g)(y)]^p} dy \leq c \int_{\mathbb{R}^n} f(y)^p dy, \]
for all nonnegative functions $f$, $g$. Then c) follows immediately from (4.6) after an application of the generalized Hölder’s inequality (4.2)
\[ M(fg)(y) \leq M_B f(y) M_B g(y) \quad y \in \mathbb{R}^n. \]

To prove that c) implies a) we test (4.7) with $f = u = \chi_Q(0,1)$ where $Q(0,1)$ is the cube centered at the origin and with radius 1:
\[ \int_{\mathbb{R}^n} M f(y)^p \frac{1}{[M_B(f)(y)]^p} dy \leq C. \]
(4.10)

On the other hand a computation using (4.4) shows that for large $x$ that
\[ M_B(f)(x) \approx \frac{1}{B^{-1}\left(\frac{1}{|x|}\right)} \]
and we can continue (4.10) with
\[ C \geq \int_{\mathbb{R}^n} M f(y)^p \frac{1}{[M_B(f)(y)]^p} dy \geq C \int_{|y| > c} \frac{1}{|y|^p} \frac{1}{B^{-1}\left(\frac{1}{|y|}\right)} dy = \]
\[ = C \int_c^\infty \frac{1}{r^{np}} \frac{1}{B^{-1}\left(\frac{1}{r}\right)} r^n \frac{dr}{t^p} \approx \int_c^\infty \frac{B(t)}{t^p} \frac{dt}{t}. \]
This shows that $c) \Rightarrow a)$.

We conclude the Chapter by proving a weighted inequality “dual” to the classical Fefferman–Stein inequality

$$\int_{\mathbb{R}^n} Mf(y)^p w(y)dy \leq c \int_{\mathbb{R}^n} f(y)^p Mw(y)dy.$$  

The main interest follows from the fact that its natural “dual” inequality, thinking as if $M$ were linear and selfadjoint, would be

$$\int_{\mathbb{R}^n} Mf(y)^p' Mw(y)^{1-p'} dy \leq c \int_{\mathbb{R}^n} f(y)^p' w(y)^{1-p'} dy.$$  

However this estimate is false in general as can be shown by taking $f = w$ positive and integrable. However we the following sharp result.

**Corollary 4.2.3**

$$\int_{\mathbb{R}^n} Mf(y)^p [M^{[p']}^{1+1}(w)(y)]^{1-p} dy \leq c \int_{\mathbb{R}^n} |f(y)|^p w(y)^{1-p} dy. \quad (4.11)$$

The proof is a consequence of (4.7), choosing $B$ such that $\bar{B} \approx t^{p'} (\log t)^{p'-1+\delta}$, combined with Lemma 5.4.1 below. See [P1] for details.
Chapter 5

Fractional integral operators

We have already discussed in Chapter 2 the relationship of a function \( f \) and its gradient via the fractional integral. Indeed, recall that if \( f \) is a sufficiently smooth function, then there is a universal constant \( C \) such that for each cube and for each \( x \in Q \)

\[
|f(x) - f_Q| \leq C I_1(|\nabla f| \chi_Q)(x)
\]

(5.1)

If we further assume that \( f \) has compact support then this inequality clearly shows that for each \( x \in \mathbb{R}^n \)

\[
|f(x)| \leq C I_1(|\nabla f|)(x).
\]

(5.2)

Recall that \( I_\alpha, 0 < \alpha < n \), denotes the fractional integral of order \( \alpha \) on \( \mathbb{R}^n \) or Riesz potentials defined by

\[
I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.
\]

5.1 Weak implies strong

The main point of this section is to illustrate the fact that a weak type inequality with a gradient (or gradient like) operator “on the right hand side” implies the corresponding strong type inequality. This is never the case in the standard interpolation theory. As we mentioned in Section 2.5 the basic ideas are taken from [LN].

Theorem 5.1.1 (weighted isoperimetric inequality) Let \( n > 1 \) and let \( \mu \) be any nonnegative measure. Then there is a constant \( C \) such that for any Lipschitz function \( f \) with compact support

\[
\left( \int_{\mathbb{R}^n} |f(x)|^{n'} d\mu \right)^{1/n'} \leq C \int_{\mathbb{R}^n} |\nabla f(x)| (M\mu(x))^{1/n'} dx.
\]

(5.3)

This result and generalizations can be found in [FPW2].
Proof: The key point is to prove a corresponding weak type estimate which in this case is
\[ \|f\|_{L^{n',\infty}(\mu)} \leq C \int_{\mathbb{R}^n} |\nabla f(x)| (M\mu(x))^{1/n'} \, dx, \] (5.4)
where we recall that
\[ \|f\|_{L^{q,\infty}(w)} = \sup_{t>0} t \, w(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/q}, \]
is the standard notation for the Marcinkiewicz “norm”. When \( q > 1 \), \( \|f\|_{L^{q,\infty}(w)} \) is equivalent to a norm, and therefore using (5.2) and Minkowski’s inequality we obtain
\[ \|f\|_{L^{n',\infty}(w)} \leq C \int_{\mathbb{R}^n} |\nabla f(x)| \| \cdot -x \|_{L^{n',\infty}(w)}^{1-n} \, dx. \]
Now observe that
\[ \| \cdot -x \|_{L^{n',\infty}(\mu)}^{1-n} = \sup \lambda \mu(\{y \in \mathbb{R}^n : |y - x|^n > \lambda\})^{1/n'} \]
and
\[ = \left( \sup \lambda \mu(\{y \in \mathbb{R}^n : |y - x|^n < \frac{1}{\lambda}\}) \right)^{1/n'} = \left( \sup_{t>0} \frac{1}{t^n} \int_{B(x,t)} \mu \right)^{1/n'} \approx (M\mu(x))^{1/n'}, \]
from which we get the desired inequality (5.5).

Now that we have the weak type estimate we proceed as follows. Given a nonnegative function \( g \) and a positive number \( \lambda \), we denote by \( \tau_\lambda(g) \) the truncation of \( g \)
\[ \tau_\lambda(g) = \min\{g, 2\lambda\} - \min\{g, \lambda\} = \begin{cases} 
0 & \text{if } g(x) \leq \lambda \\
\lambda & \text{if } g(x) < 2\lambda \\
g(x) - \lambda & \text{if } \lambda \leq g(x) < 2\lambda 
\end{cases} \]
A very nice fact about Lipschitz functions is that they are preserved by absolute values and by truncations. Hence if \( f \in \Lambda(1) \Rightarrow \tau_\lambda(\{|f|\}) \in \Lambda(1), \lambda > 0 \). Now, if we define for each integer \( k \), \( F_k = \{x \in \mathbb{R}^n : 2^k < |f(x)| \leq 2^{k+1}\} \), we have
\[ \int_{\mathbb{R}^n} |f(x)|^{n'} w(x) \, dx = \sum_{k=-\infty}^{\infty} \int_{x \in \mathbb{R}^n, 2^k < |f(x)| \leq 2^{k+1}} |f(x)|^{n'} w(x) \, dx \]
\[ \approx \sum_{k=-\infty}^{\infty} 2^{kn'} w(F_{k+1}) \]
Now, observe that for \( x \in F_{k+1}, 2^k = \tau_2^k(|f|)(x) \). Also observe that the support of the gradient of \( \tau_2^k(|f|) \) equals \( F_k \). Then by the weak type estimate
\[ \left( \int_{\mathbb{R}^n} |f(x)|^{n'} \, d\mu(x) \right)^{1/n'} \leq C \sum_{k=-\infty}^{\infty} 2^k \mu(F_{k+1})^{1/n'} \]
\[
\leq C \sum_{k=-\infty}^{\infty} 2^k \mu\{x \in \mathbb{R}^n : \tau_{2^k}(|f|)(x) > 2^{k-1}\} \leq C \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} |\nabla \tau_{2^k}(|f|)(x)| (M \mu(x))^{1/n'} dx
\]
\[
= C \sum_{k=-\infty}^{\infty} \int_{F_k} |\nabla f(x)| (M \mu(x))^{1/n'} dx = C \int_{\mathbb{R}^n} |\nabla f(x)| (M \mu(x))^{1/n'} dx
\]
since the sets \(F_k\) are pairwise disjoint.

Remark 5.1.2 In fact the following estimate holds
\[
\left\| f \right\|_{L^{n'/1}(\mu)} \leq C \int_{\mathbb{R}^n} |\nabla f(x)| (M \mu(x))^{1/n'} dx, \quad (5.5)
\]
To prove sharp estimates for fractional integrals are much more difficult as we will see now.

Theorem 5.1.3 Suppose that \(0 < \alpha < n\) and that \(v\) is a nonnegative measurable function on \(\mathbb{R}^n\).

Let \(1 < p < \infty\). Then there exists a constant \(C = C_{n,p}\) such that
\[
\int_{\mathbb{R}^n} |I_\alpha f(x)|^p v(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M_{\alpha p}(M^{[p]} v)(x) dx. \quad (5.6)
\]
This result shows the \(M_{\alpha p}(M^{[p]} v)\) is the operator that "controls" fractional integrals in the \(L^p\) norm. It should be compared with (1.4).

This result can be found in [P4] and can be seen as a sharpening version of the Chang-Wilson-Wolff estimate derived in [CWW] which in turn is sharper than the condition of Fefferman-Phong related to the theory of Schrödinger operators (see [P4] for more details).

Proof: By the duality between the spaces \(L^p(\mathbb{R}^n)\) and \(L^{p'}(\mathbb{R}^n)\), it is enough to show that there exists a positive constant \(C\) such that
\[
(I) = \int_{\mathbb{R}^n} I_\alpha f(x) v(x) g(x) dx \leq C \left[ \int_{\mathbb{R}^n} f(x)^p M_{\alpha p}(M^{[p]} v)(x) dx \right]^{1/p} \left[ \int_{\mathbb{R}^n} g(x)^{p'} dx \right]^{1/p'}.
\]
Furthermore, by density, we can assume that both functions \(f\) and \(g\) are nonnegative, bounded and have compact support.

5.2 Step 1: Discretization of the potential operator

In this section we will show a procedure to discretize the operator which is interesting on its own right. In fact the method can be used to give a different proof which avoids
good–lambda inequalities of a well–known result of Muckenhoupt and Wheeden [MW]. See also [PW] for further results related to subelliptic vector fields.

Observe first that \( I_\alpha f(x) < \infty \) for all \( x \in \mathbb{R}^n \) so that the manipulations that we are about to start make sense.

We discretize the operator \( I_\alpha f \) as follows:

\[
I_\alpha f(x) = \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}} \chi_Q(x) \int_{\ell(Q)/2 < |x-y| \leq \ell(Q)} \frac{f(y)}{|x-y|^{n-\alpha}} dy
\]

\[
\leq C \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/n}}{|Q|} \int_{|y-x| \leq \ell(Q)} f(y) dy \chi_Q(x).
\]

\( \mathcal{D} \) denotes the family of dyadic cubes on \( \mathbb{R}^n \). Since the ball \( B(x, \ell(Q)) \) is contained in the cube \( 3Q \) when \( x \in Q \) we have

\[
I_\alpha f(x) \leq \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/n}}{|Q|} \int_{3Q} f(y) dy \chi_Q(x),
\]

and then

\[
\int_{\mathbb{R}^n} I_\alpha f(x) g(x) d\mu(x) \leq c \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/n}}{|Q|} \int_{3Q} f(y) dy \int_Q g(x) d\mu(x).
\]

(5.7)

### 5.3 Step 2: Replacing dyadic cubes by dyadic Calderón–Zygmund cubes

Observe that the sum in equation (5.7) runs over all the dyadic cubes from \( R^n \) and this is too much to get a good control of, therefore the next task is to replace the family \( \mathcal{D} \) by a subclass that satisfies certain good properties. This family is formed by considering appropriate Calderon-Zygmund cubes at each level \( a^k \) where \( a \) is a large enough constant.

To be more precise we have the following lemma.

**Lemma 5.3.1** Let \( a > 2^n \), then there exists a family of cubes \( \{Q_{k,j}\} \) and a family of pairwise disjoint subsets \( \{E_{k,j}\} \), with \( E_{k,j} \subset Q_{k,j} \) such that for each \( k, j \),

\[
|Q_{k,j}| < \frac{1}{1 - \frac{a}{2^n}} |E_{k,j}|
\]

and such that

\[
\int_{\mathbb{R}^n} I_\alpha f(x) g(x) d\mu(x) \leq C \sum_{k,j} \frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|} \int_{3Q_{k,j}} f(y) dy \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} g(y) d\mu(y) |E_{k,j}|.
\]

(5.9)
**Proof:** Recall that $a > 2^n$ and consider for each integer $k$ the set

$$D_k = \{x \in \mathbb{R}^n : M^d(g \, d\mu)(x) > a^k\},$$

where $M^d$ is the usual dyadic Hardy–Littlewood maximal operator. Then we apply the Calderón–Zygmund covering lemma (see Appendix I) it follows that if $D_k$ is not empty there exists some dyadic cube $Q$ with

$$a^k < \frac{1}{|Q|} \int_Q g(y) \, d\mu(y), \quad (5.10)$$

then $Q$ is contained in one dyadic cube satisfying this condition and maximal with respect to inclusion. Thus for each $k$ we can write $D_k = \bigcup Q_{k,j}$ where the cubes $\{Q_{k,j}\}$ are nonoverlapping, they satisfy (5.10), and by maximality we also get

$$a^k < \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} g(y) \, d\mu(y) \leq 2^n a^k. \quad (5.11)$$

We also need the following property. For all integers $k, j$ we let $E_{k,j} = Q_{k,j} \setminus D_{k+1}$. Then $\{E_{k,j}\}$ is a disjoint family of sets which satisfy

$$|Q_{k,j} \cap D_{k+1}| < \frac{2^n}{a} |Q_{k,j}|, \quad (5.12)$$

and therefore

$$|Q_{k,j}| < \frac{1}{1 - \frac{2^n}{a}} |E_{k,j}|. \quad (5.13)$$

Indeed, by standard properties of the dyadic cubes we can compute what portion of $Q_{k,j}$ is covered by $D_{k+1}$ as in [GCRdF] p. 398:

$$\frac{|Q_{k,j} \cap D_{k+1}|}{|Q_{k,j}|} = \sum_i \frac{|Q_{k,j} \cap Q_{k+1,i}|}{|Q_{k,j}|} = \sum_{i : Q_{k+1,i} \subset Q_{k,j}} \frac{|Q_{k+1,i}|}{|Q_{k,j}|} \leq \frac{a^k}{a^{k+1}} \sum_{i : Q_{k+1,i} \subset Q_{k,j}} \frac{1}{|Q_{k,j}|} \int_{Q_{k+1,i}} g(y) \, d\mu(y) \leq \frac{a^k}{a^{k+1}} \frac{1}{|Q_{k,j}|} \int_{Q_{k,j} \cap \bigcup_{Q_{k+1,i}}} g(y) \, d\mu(y) \leq \frac{2^n}{a},$$

by the right hand side of (5.11). This gives (5.12).

We continue with the proof of the Lemma by adapting some ideas from [SW]. For each integer $k$ we let

$$C^k = \{Q \in \mathcal{D} : a^k < \frac{1}{|Q|} \int_Q g(y) \, d\mu(y) \leq a^{k+1}\}.$$
Every dyadic cube $Q$ for which $\int_Q g(y) \, d\mu(y) \neq 0$ belongs to exactly one $C^k$. Furthermore, if $Q \in C^k$ it follows that $Q \subset Q_{k,j}$ for some $j$. Then

$$\int_{\mathbb{R}^n} I_\alpha f(x) \, g(x) \, d\mu(x)$$

$$\leq \sum_{k \in \mathbb{Z}} \sum_{Q \in C^k} \frac{|Q|^{\alpha/n}}{|Q|} \int_{3Q} f(y) \, dy \int_Q g(y) \, d\mu(y)$$

$$\leq a \sum_{k \in \mathbb{Z}} a^k \sum_{j \in \mathbb{Z}} \sum_{Q \in C^k} |Q|^{\alpha/n} \int_{3Q} f(y) \, dy$$

$$\leq C \sum_{k,j} \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} g(y) \, d\mu(y) \left| Q_{k,j} \right|^{\alpha/n} \int_{3Q_{k,j}} f(y) \, dy$$

The last inequality follows from the fact that for each dyadic cube $P$

$$\sum_{Q \in D, Q \subset P} |Q|^\alpha/n \int_{3Q} f(y) \, dy \leq C |P|^\alpha/n \int_{3P} f(y) \, dy$$

Indeed, if $\ell(P) = 2^{-\nu_0}$

$$\sum_{Q \in D, Q \subset P} |Q|^\alpha/n \int_{3Q} f(y) \, dy = C \sum_{\nu \geq \nu_0} \sum_{Q \in D, Q \subset P, \ell(Q) = 2^{-\nu}} 2^{-\nu \alpha} \int_{3Q} f(y) \, dy$$

$$= C \sum_{\nu \geq \nu_0} 2^{-\nu \alpha} \sum_{Q \in D, Q \subset P, \ell(Q) = 2^{-\nu}} \int_{3Q} f(y) \, dy$$

$$\leq C |P|^\alpha/n \sum_{Q \in D, Q \subset P, \ell(Q) = 2^{-\nu}} \int_{3Q} f(y) \, dy \leq C |P|^\alpha/n \int_{3P} f(y) \, dy$$

since the overlap is finite. Hence

$$\int_{\mathbb{R}^n} I_\alpha f(x) \, g(x) \, d\mu(x)$$

$$\leq C \sum_{k,j} \left| \frac{3Q_{k,j}}{|3Q_{k,j}|} \right|^{\alpha/n} \int_{3Q_{k,j}} f(y) \, dy \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} g(y) \, d\mu(y) \left| Q_{k,j} \right|$$

$$\leq C \sum_{k,j} \left| \frac{3Q_{k,j}}{|3Q_{k,j}|} \right|^{\alpha/n} \int_{3Q_{k,j}} f(y) \, dy \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} g(y) \, d\mu(y) \left| E_{k,j} \right|$$

by (5.13).
5.4 Step 3: Patching the pieces together

Now, by Lemma 5.3.1 with \(d\mu(x) = v(x)^{1/p} \, dx\) we have that

\[
(I) \leq C \sum_{k,j} \frac{|Q_{k,j}|^{\alpha/n}}{|3Q_{k,j}|} \int_{3Q_{k,j}} f(y) \, dy \int_{Q_{k,j}} v(y)^{1/p} g(y) \, dy \left|E_{k,j}\right|.
\]

Next we apply the generalized Hölder’s inequality (4.2) with respect to the associated the spaces \(L^p(\log L)^{(p-1)(1+\epsilon)}\) and \(L^{p'}(\log L)^{-1-\epsilon}\), \(\epsilon > 0\), (see [O] for instance). After that we apply Hölder’s inequality at the discrete level with exponents \(p\) and \(p'\). We can follow the estimate with

\[
C \sum_{k,j} \frac{|Q_{k,j}|^{\alpha/n}}{|3Q_{k,j}|} \int_{3Q_{k,j}} f(y) \, dy \left\|v^{1/p}\right\|_{L^p(\log L)^{(p-1)(1+\epsilon)}} \left|E_{k,j}\right|^{1/p} \left\|g\right\|_{L^{p'}(\log L)^{-1-\epsilon}} \left|E_{k,j}\right|^{1/p'}
\]

\[
\leq C \left[ \sum_{k,j} \left( \frac{|Q_{k,j}|^{\alpha/n}}{|3Q_{k,j}|} \int_{3Q_{k,j}} f(y) \, dy \right)^p \left\|v^{1/p}\right\|^p_{L^p(\log L)^{(p-1)(1+\epsilon)}} \left|E_{k,j}\right| \right]^{1/p} \times \left[ \sum_{k,j} \left\|g\right\|_{L^{p'}(\log L)^{-1-\epsilon}} \left|E_{k,j}\right| \right]^{1/p'}.
\]

Now, if \(\epsilon = \frac{|p|}{p-1} - 1 > 0\), then

\[
\left\|v^{1/p}\right\|^p_{L^p(\log L)^{(p-1)(1+\epsilon)}} = \left\|v\right\|^p_{L(\log L)^{|p|}} \tag{5.14}
\]

Now, we need the following Lemma

**Lemma 5.4.1** If \(k = 1, 2, 3, \cdots\), then there exists a constant \(C = C_n\) such that for all bounded functions \(f\) with support contained in \(Q\)

\[
\|f\|_{L(\log L)^k, Q} \leq C \left|Q\right| \int_Q M^k f(y) \, dy. \tag{5.15}
\]

**Proof:** By definition of Luxemburg norm (5.15) is equivalent to showing that for some positive constant \(c\)

\[
\frac{1}{|Q|} \int_Q \frac{w}{c \lambda_Q} \log^k (1 + \frac{w}{c \lambda_Q}) \leq 1.
\]

where we denote \(\lambda_Q = \frac{1}{|Q|} \int_Q M^k w\).

We first recall the following Wiener estimate which can be seen as a “reverse” weak type \((1,1)\) estimate (cf. (4.8 with \(B(t)=t\)) which can be deduced from Appendix 7. For each \(t > \int_Q\)

\[
\frac{1}{t} \int_{\{x \in Q : f(x) > t\}} f(x) \, dx \leq 2^n \left|\{x \in Q : M^d_{Q}(f)(x) > t\}\right| \tag{5.16}
\]

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To prove the lemma we use induction. We prove (5.15) with $k=1$ and let $f = \frac{w}{2^k \lambda_Q}$.

Now, using the formula
\[
\int_X \phi(f) \, d\nu = \int_0^{\infty} \phi'(t) \nu(\{x \in X : f(x) > t\}) \, dt
\]
we have
\[
\frac{1}{|Q|} \int_Q f(x) \log(1 + f(x)) \, dx = \frac{1}{|Q|} \int_0^{\infty} \frac{1}{1 + t} f(\{x \in Q : f(x) > t\}) \, dt
\]
\[
= \frac{1}{|Q|} \int_0^1 \frac{1}{1 + t} f(\{x \in Q : f(x) > t\}) \, dt + \frac{1}{|Q|} \int_1^{\infty} \frac{1}{1 + t} f(\{x \in Q : f(x) > t\}) \, dt = I + II
\]
Now, recalling that $\lambda_Q = \frac{1}{|Q|} \int_Q Mw$ we observe that $I \leq 1$ since
\[
\frac{f(Q)}{|Q|} = \frac{1}{|Q|} \int_Q \frac{w}{c |Q|} \int_Q Mw \leq 1
\]
by the of $M$ and since $c \geq 1$. For $II$ we use estimate (5.16):
\[
II \leq \frac{1}{|Q|} \int_1^{\infty} \frac{1}{1 + t} f(\{x \in Q : f(x) > t\}) \, dt \leq \frac{2^n}{|Q|} \int_1^{\infty} \frac{t}{1 + t} \nu(\{x \in Q : Mf(x) > t\}) \, dt
\]
\[
\leq \frac{2^n}{|Q|} \int_0^{\infty} \nu(\{x \in Q : Mf(x) > t\}) \, dt = \frac{2^n}{|Q|} \int_Q Mf(x) \, dx = \frac{2^n}{|Q|} \int_Q Mw(x) \, dx = \frac{1}{2^n \lambda_Q} = 1.
\]

We now assume that the estimate holds for certain $k$. Then with $f = \frac{w}{c \lambda_Q}$ and $\lambda_Q = \frac{1}{|Q|} \int_Q M^{k+1} w$.
\[
\frac{1}{|Q|} \int_Q f \log^{k+1}(1 + f) \, dx = \frac{k + 1}{|Q|} \int_0^{\infty} \frac{\log^{k}(1 + t)}{1 + t} f(\{x \in Q : f(x) > t\}) \, dt
\]
\[
= \frac{k + 1}{|Q|} \left( \int_0^1 + \int_1^{\infty} \right) \frac{\log^{k}(1 + t)}{1 + t} f(\{x \in Q : f(x) > t\}) \, dt = I + II
\]
Again by the Lebesgue differentiation theorem we can chose $c = c_k \geq 1$ such that $I \leq 1$.

Let $\phi(t) = t \log^k(1 + t)$. Then $\phi'(t) = \log^k(1 + t) + \frac{kt \log^{k-1}(1 + t)}{1 + t}$ and again for $II$ we use estimate (5.16):
\[
II \leq \frac{k + 1}{|Q|} \int_1^{\infty} \frac{\log^{k}(1 + t)}{1 + t} f(\{x \in Q : f(x) > t\}) \, dt
\]
\[
\leq \frac{2^n (1 + k)}{|Q|} \int_1^{\infty} \frac{t \log^{k}(1 + t)}{1 + t} \nu(\{x \in Q : Mf(x) > t\}) \, dt
\]
\[
\leq \frac{2^n(1 + k)}{|Q|} \int_1^\infty \varphi'(t) \{ x \in B : Mf(x) > t \} \, dt \\
\leq \frac{2^n(1 + k)}{|Q|} \int_Q Mf(x) \log^k(1 + Mf(x)) \, dx \leq 1.
\]

In the last inequality we have used the induction hypothesis since \( f = \frac{w}{c\lambda_Q} \) and since
\[
\lambda_Q = \frac{1}{|Q|} \int_Q M^{k+1}w \geq \frac{1}{|Q|} \int_Q M^k w, \text{ for appropriate constant } c. \]
This completes the proof of the lemma and hence the proof of Theorem 5.1.3.

\[
\square
\]

Then using (5.14) and (5.15) we get

\[
(I) \leq C \left[ \sum_{k,j} \left( \frac{|Q_{k,j}|^{a/n}}{|3Q_{k,j}|} \int_{3Q_{k,j}} f(y) \, dy \right)^p \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} M^{[p]v}(y) \, dy \right]^{1/p} \times \\
\times \left[ \sum_{k,j} \left( \int_{E_{k,j}} M^{[p]v}(y) \, dy \right)^p \right]^{1/p'} \times \\
\leq C \left[ \sum_{k,j} \left( \frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} f(y) M_{ap}(M^{[p]v})(y)^{1/p} \, dy \right)^p \right]^{1/p} \times \\
\times \left[ \sum_{k,j} \int_{E_{k,j}} M^{[p]v}(y) \, dy \right]^{1/p'} \times \\
\leq C \left[ \sum_{k,j} \int_{E_{k,j}} M\left( M^{[p]v}\right)^{1/p}(x)^p \, dx \right]^{1/p} \times \left[ \int_{\mathbb{R}^n} M^{[p]v}(x)^p \, dx \right]^{1/p'} \times \\
\approx C \left[ \int_{\mathbb{R}^n} f(x)^p M_{ap}(M^{[p]v})(x) \, dx \right]^{1/p} \times \left[ \int_{\mathbb{R}^n} g(x)^p \, dx \right]^{1/p'}.
\]

We used first that the sets in the family \( E_{k,j} \) are pairwise disjoint. Also it is used that \( M \) is a bounded operator on \( L^p(\mathbb{R}^n) \) and that the maximal operator \( M^{[p]v} \) is bounded on \( L^{p'}(\mathbb{R}^n) \) by Lemma 4.2.2 since the Young function \( B(t) \approx t^{p'}(\log t)^{-1-\epsilon} \) satisfies the \( B_{p'} \) condition.
Chapter 6

Singular Integral Operators

6.1 Introduction: Singular Integral Operators and smoothness of functions

What follows can be seen as a motivation to the study of weighted inequalities for singular integrals.

We fix a pair of weights \( u, v \) and consider the weighted inequality

\[
\left( \int_{\mathbb{R}^n} |f|^q u \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |\nabla f|^p v \right)^{1/p}.
\] (6.1)

It is easy to see that (6.1) is implied by

\[
\left( \int_{\mathbb{R}^n} |I_1 f|^q u \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f|^p v \right)^{1/p}.
\] (6.2)

since for appropriate functions (see the introduction to Chapter 5) we have

\[ |f(x)| \leq C I_1(|\nabla f|)(x). \]

Inequality (6.2) should be, in principle, be more flexible to be treated due to all the information we have about fractional integrals. The question is whether these inequalities are equivalent. The answer is no in general. Indeed, assuming that (6.1) holds we test it with \( f = I_1(g) \) were \( g \) is any smooth function with compact support. Then we are led to consider \( \nabla(I_1(g)) \) and it is not hard to show that

\[ \nabla(I_1(f)) = c R g \]

where \( R = \{R_j g\}_{j=1}^n \) is the classical vector-valued Riesz transform where each component is defined by

\[
R_j g(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{x_j - y_j}{|x-y|^{n+1}} g(y) \, dy
\]
Therefore we are led to study of weighted norm inequalities for the Riesz Transforms, namely
\[ \left( \int_{\mathbb{R}^n} |Rf|^p v \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} |f|^p v \right)^{1/p}. \]

It is possible to show that a necessary and sufficient condition is the $A_p$ condition. In fact we will show in Corollary 6.2.4 that the $A_p$ condition is also a sufficient condition for a wider class of operators. To simplify the presentation we can think of the case of regular convolution singular integrals. These operators are defined for smooth functions $f$ with compact support by
\[ Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x-y) f(y) \, dy. \]

We will assume that:

a) $T$ is bounded on $L^2(\mathbb{R}^n)$ or what is equivalent the Fourier transform of $K$, as a principal value distribution, is a bounded function.

b) There is a constant $c$ such that for every $x$ different from zero:
\[ |K(x)| \leq \frac{c}{|x|^n} \]

c) $K$ satisfies regularity condition: there is a constant $C$ such that for $|x| > 2|y|$
\[ |K(x-y) - K(x)| \leq C \frac{|y|}{|x|^{n+1}}. \]

This condition is satisfied whenever
\[ |\nabla K(x)| \leq \frac{c}{|x|^{n+1}} \]
when $x$ is different from zero.

We remit the reader to the books [Ch], [Jo] or [D] for further information and generalizations about singular integrals.

### 6.2 Weighted $L^p$ estimates

The basic tool we are going to use is a modification of the sharp maximal operator $M^\#$ of C. Fefferman and E. Stein: for $\delta > 0$ we define the $\delta$–sharp maximal operator $M^\#_\delta$ as
\[ M^\#_\delta(f) = M^\#(|f^\delta|^{1/\delta}). \]

Recall that $M^\#$ is defined by
\[ M^\# f(x) = \sup_{x \in Q} \inf_{c} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy \approx \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy. \]

Our first result concerns operators that are controlled in some sense by the Hardy–Litllewood maximal function. To be more precise we give the following definition.
Definition 6.2.1 We say that $T$ an operator satisfies the condition (D) if for any $0 < \delta < 1$ there is a constant $c$ such that for all $f$,

$$M_\delta^\#(Tf)(x) \leq cMf(x), \quad (D).$$

In the most classical situation this pointwise estimate is derived for Calderón-Zygmund operators with $\delta = 1$ and $Mf$ is replaced by $M(f^r)^{1/r}$ for all $r > 1$. However this estimate is not sharp enough for the results we will present below.

Condition (D) is used in [AP] to derive a different proof of Coifman’s theorem [C] relating the (weighted) $L^p$ norm of Calderón-Zygmund operators and (weighted) $L^p$ norm of the Hardy–Littlewood maximal function. See Theorem 6.2.3 below.

Some examples of operators satisfying condition (D) are:

- **Calderón-Zygmund operators** These operators are generalization of the regular singular integral operators as defined above. See [Jo] [Ch] or [D] for appropriate definitions. They are the model examples of operators satisfying condition (D). This was observed in [AP] (cf Lemma 6.2.2 below).

- **weakly strongly singular integral operators** These operators were considered by C. Fefferman in [F].

- **pseudo-differential operators.** To be more precise, the pseudo-differential operators satisfying condition (D) are those that belong to the Hörmander class (see Hörmander [H]).

- **oscillatory integral operators** These operators were by introduced by Phong and Stein [PS].

The proof of the last three cases can be found in [AHP].

We illustrate how to verify condition (D) in the case of regular singular integral operators.

**Lemma 6.2.2** Let $T$ be any regular singular integral operator or more generally any Calderón-Zygmund operator. Let $0 < \delta < 1$. Then, there exists a positive constant $c = c_\delta$ such that,

$$M_\delta^\#(Tf)(x) \leq cMf(x)$$

for all smooth functions $f$.

**Proof:** Let $Q = Q(x, r)$ be an arbitrary cube. Since $0 < \delta < 1$ implies $|\alpha|^{\delta} - |\beta|^{\delta} \leq |\alpha - \beta|^\delta$ for $\alpha, \beta \in \mathbb{R}$ it is enough to show for some constant $c = c_Q$ that there exists $C = C_\delta > 0$ such that,

$$\left( \frac{1}{|Q|} \int_Q |Tf(y) - c|^{\delta} \, dy \right)^{1/\delta} \leq C Mf(x). \quad (6.3)$$
Let $f = f_1 + f_2$, where $f_1 = f \chi_{2Q}$. If we pick $c = (T(f_2))_Q$, we can estimate the left hand side of (6.3) by a multiple of

$$\left( \frac{1}{|Q|} \int_Q |T(f_1)(y)|^\delta \, dy \right)^{1/\delta} + \left( \frac{1}{|Q|} \int_Q |T(f_2) - (T(f_2))_Q|^\delta \, dy \right)^{1/\delta} = I + II.$$

Since $T : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ ([Ch] or [Jo]) and $0 < \delta < 1$, Kolmogorov’s inequality (2.25) yields

$$I \leq C \int_{\mathbb{R}^n} |f_1(y)| \, dy = C \int_{2Q} |f(y)| \, dy \leq C \, M(f)(x).$$

To take care of $II$ we use the regularity of the kernel. Indeed, by Jensen’s inequality and Fubini’s theorem we have

$$II \leq \frac{1}{|Q|} \int_Q |T(f_2)(y) - (T(f_2))_Q| \, dy \leq$$

$$\leq \frac{1}{|Q|^2} \int_Q \int_Q \int_{\mathbb{R}^n - 2Q} |k(y - w) - k(z - w)||f(w)| \, dw \, dz \, dy \leq$$

$$\leq \frac{1}{|Q|^2} \int_Q \int_Q \sum_{j=1}^\infty \int_{2j+1} \frac{|y - z|}{|x - w|^{n+1}} |f(w)| \, dw \, dz \, dy \leq$$

$$\leq C \sum_{j=1}^\infty \frac{2^{-j}}{(2j+1)^n} \int_{2j+1Q} |f(w)| \, dw \leq C \sum_{j=1}^\infty 2^{-j} Mf(x) = C \, Mf(x).$$

The following a priori estimate for the operator has many applications as we will see below.

**Theorem 6.2.3** Let $T$ be a operator satisfying condition (D). Also we let $0 < p < \infty$ and suppose that $w \in A_\infty$. Then the following inequalities hold

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x) \, dx \leq C \int_{\mathbb{R}^n} Mf(x)^p \, w(x) \, dx,$$  \hspace{1cm} (6.4)

and

$$\sup_{\lambda > 0} \lambda^p w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq C \, [w]_{A_\infty}^p \sup_{\lambda > 0} \lambda^p w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}),$$  \hspace{1cm} (6.5)

whenever $f$ is a function for which the left hand side is finite, where the constant $C$ always depend upon the $A_\infty$ constant of $w$.  

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\textbf{Proof:} By the Lebesgue differentiation theorem and Fefferman-Stein inequality (2.30)

\[
\int_{\mathbb{R}^n} |Tf|^p \, w \leq \int_{\mathbb{R}^n} M(|Tf|^\delta)^{p/\delta} \, w \leq C \int_{\mathbb{R}^n} M^\#(|Tf|^\delta)^{p/\delta} \, w
\]

\[
= C \int_{\mathbb{R}^n} M^\#(Tf)^p \, w \leq C \int_{\mathbb{R}^n} (Mf)^p \, w
\]

since $T$ satisfies the condition (D).

The other inequality is proved in the same way. \qed

\textbf{Corollary 6.2.4} Let $T$ be a operator satisfying condition (D).

a) Let $1 < p < \infty$ and suppose that $w \in A_p$. Then, there exists a constant $C$ such that

\[
\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \, w(x) \, dx.
\] (6.6)

b) Suppose that $w \in A_1$. Then, there exists a constant $C$ such that for all functions $\lambda$

\[
w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \, w(x) \, dx
\] (6.7)

Observe that we have imposed appropriate conditions on the weights to get the desired estimates. In next theorem we don’t impose any condition on the weight $w$.

\textbf{Corollary 6.2.5} Let $T$ be a linear operator such that its conjugate operator satisfies condition (D). Let $1 < p < \infty$ and let $w$ be a weight. Then

\[
\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \, M^{[p]+1}w(x) \, dx.
\] (6.8)

Furthermore:

a) the inequality does not hold if $M^{[p]+1}$ is replaced by the smaller operator $M^{[p]}$.

b) the operator $M^{[p]+1}$ can be replaced by the smaller and sharp operators

\[
M_{L(\log L)^{p-1+\epsilon}}
\]

$\epsilon > 0$, being false the result for the case $\epsilon = 0$.

\textbf{Remark 6.2.6} We remark that the unweighted theory or the one weight $A_p$ theory coincides for both the Hardy–Littlewood maximal function and singular integrals. However this is not the case when considering the two weight theory. This phenomenon is well reflected in last corollary, result that can be found in [P2].

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It should be compared estimate (6.8) with the one related to the maximal function (1.4) observing that by the Lebesgue differentiation theorem we always have

\[ w \leq Mw \leq M^2w \leq M^3w \leq \cdots \]

We just point out in applications most of the operators (such as those listed above are either selfadjoint or the adjoint is a constant multiple of the operator).

The proof of the corollary is just a combination of Theorem 6.2.4, Corollary 4.2.3 and the easy part of the factorization theorem. Indeed, by duality (6.8) is equivalent to

\[
\int_{\mathbb{R}^n} |Tf(x)|^{p'} [M^{[p]+1}(w)(x)]^{1-p'} dx \leq c \int_{\mathbb{R}^n} |f(x)|^{p'} w(x)^{1-p'} dx.
\]

We claim that \([Mf(x)]^{1-q}, q > 1,\) in an \(A_\infty\). In fact We claim that \([Mf(x)]^{1-q} \in A_{q+\epsilon},\) \(\epsilon > 0,\) but is not in \(A_q.\) Indeed, we write

\[
(Mf(x))^{1-q} = [(Mf(x))^{1-q}] \cdot \left[ \frac{1}{q-1} \right]^{1-p}
\]

which by Lemma 1.0.7 and the easy part of the factorization theorem yields the claim. Therefore by Theorem 6.2.4 and Corollary 4.2.3 we have

\[
\int_{\mathbb{R}^n} |Tf(x)|^{p'} [M^{[p]+1}(w)(x)]^{1-p'} dx \leq c \int_{\mathbb{R}^n} (Mf(x))^{p'} [M^{[p]+1}(w)(x)]^{1-p'} dx
\]

\[
\leq c \int_{\mathbb{R}^n} |f(x)|^{p'} w(x)^{1-p'} dx.
\]

In the case \(p = 1\) we have the following result from [P2]. We will use the following Young function \(A_\epsilon(t) = t \log(1 + t)^\epsilon,\) \(\epsilon > 0.\)

**Theorem 6.2.7** Let \(T\) be a linear operator such that its conjugate operator satisfies condition (D). Then for any weight \(w, \epsilon > 0\) and function \(f,\)

\[ w\left( \{ x \in \mathbb{R}^n : |Tf(x)| > t \} \right) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f| M_{L(log L)^\epsilon} \ w \ dx. \]  \[(6.9)\]

We conjecture that this result must hold for \(\epsilon = 0\) but we don’t know how to prove it. This result must also be compared with (1.3).

To conclude this section we just mention that (6.9) is the key inequality in some recent work [CP1] where some sharp two weight, weak-type norm inequalities for singular integral operators are derived. We just state the main result.

**Theorem 6.2.8** Let \(T\) be a Calderón-Zygmund operator. Given a pair of weights \((u, v)\) and \(p, 1 < p < \infty,\) suppose that for some \(\epsilon > 0\) and \(K > 0\) and for all cubes \(Q,\)

\[ \|u\|_{L_{(log L)^{-p-1+\epsilon}}} \left( \frac{1}{|Q|} \int_Q v^{1-p'} dx \right)^{p-1} \leq K. \] \[(6.10)\]
Then for every \( t > 0 \) and \( f \in L^p(v) \),

\[
u(\{ x \in \mathbb{R}^n : |Tf(x)| > t \}) \leq \frac{C}{t^p} \int_{\mathbb{R}^n} |f|^p v dx.
\]

(6.11)

Condition (6.10) is stronger than the \( A_p \) conditions for two weights, namely

\[
\left( \frac{1}{|Q|} \int_Q u(y) \, dy \right) \left( \frac{1}{|Q|} \int_Q v(y)^{1-p'} \, dy \right)^{p-1} \leq K
\]

which is a necessary and sufficient condition for the corresponding two weight problem for the Hardy–Littlewood maximal function, namely

\[
u(\{ x \in \mathbb{R}^n : Mf(x) > t \}) \leq \frac{C}{t^p} \int_{\mathbb{R}^n} |f|^p v dx.
\]

The condition of the theorem is sharp. However, for other operators such as fractional integrals or commutators, we are not able to derive similar estimates using corresponding sharp conditions as we would expect. See [CP2].

### 6.3 Banach function spaces

We have seen in the previous section the relevance of Coifman’s theorem (6.4). We will see in this section that it can also be applied to other situations, for instance when estimating operators within the context of Banach function spaces (BFS). These are interesting generalizations of the Lebesgue spaces as well other spaces such as Lorentz and Orlicz spaces. We refer to [BS] for more information about the subject.

Next result provides a different proof of the classical theorem of Boyd (see [BS] p. 154). Boyd’s proof only works for the one dimensional Hilbert transform. Our method is more flexible and allows to consider any operator satisfying condition (D). The key points are:

a) a good a priori weighted inequality \((1,1)\) for the operator; to be precise we need estimate (6.4) with \( p = 1 \) but only for any weight \( w \) in the \( A_1 \) class.

b) We need to adapt “Rubio de Francia’s algorithm” to the context of BFS.

c) To adapt this algorithm we need to use Lorentz-Shimogaki’s theorem (valid this time on \( R^n \)) which characterizes the rearrangement invariant BFS for which the Hardy–Littlewood maximal is bounded (see [BS] p. 154).

Rearrangement invariant BFS are special cases of BFS. The basic property is that if \( u \) and \( v \) are equidistributed and \( u \) belongs to a BFS then \( v \) belongs to the space and their norm coincide.

**Theorem 6.3.1** Let \( T \) be a operator satisfying condition (D) and let \( X \) be a BFS such that the Hardy–Littlewood maximal function is bounded on \( X' \), the associated space to \( X \), then

\[
\| Tf \|_X \leq c \| Mf \|_X.
\]
Therefore, if the Hardy–Littlewood maximal function is also bounded on $X$ then $T$ is bounded on $X$.

**Corollary 6.3.2** Let $T$ be a operator satisfying condition (D) and let $X$ be a rearrangement invariant BFS such that the Boyd indices $\alpha_X$ and $\overline{\alpha}_X$ satisfy

$$0 < \alpha_X \leq \overline{\alpha}_X < 1.$$  

Then $T$ is bounded on $X$.

**Proof of the Theorem:** The $\|Tf\|_X$ can be written as

$$\|Tf\|_X = \sup \left| \int_{\mathbb{R}^n} Tfg \right|$$

where the supremum is taken over all functions $g \in X'$. Fix one of these $g$ we have

$$\int_{\mathbb{R}^n} |Tf||g|$$

We now adapt Rubio de Francia’s algorithm to this context: consider

$$G = \sum_{k=0}^{\infty} \frac{M^k(g)}{(2A)^k}$$

where $M^k$ is the operator $M$ iterated $k$ times and $A$ is the norm of $M$ as bounded operator on $X'$. It is immediate to see that:

a) $g \leq G$

b) $\|G\|_{X'} \leq 2 \|g\|_{X'}$

c) $G \in A_1$, in fact $MG \leq 2AG$

In particular since $G \in A_\infty$ we can apply Theorem 6.4

$$\int_{\mathbb{R}^n} |Tf||g| \leq \int_{\mathbb{R}^n} |Tf|G \leq C \int_{\mathbb{R}^n} MfG \leq C \|Mf\|_X \|G\|_{X'} \leq C \|Mf\|_X \|g\|_{X'}.$$

Then, taking the supremum over all $g \in X'$ we deduce the theorem.

The corollary follows directly from Lorentz-Shimogaki’s characterization of the rearrangement invariant BFS for which the Hardy–Littlewood maximal is bounded as can be found in [BS].
6.4 Commutators: singular integrals with higher degree of singularity

The purpose of this section is to illustrate that the basic ideas developed in the previous sections of this chapter can be used to treat other operators of interest such as commutator of singular integral operators with \( BMO \) functions. We want to make emphasize that the method is sharp since the results obtained show that these commutators share a higher singularity as compared with Calderón–Zymund operators. This is not reflected in the known results from the literature.

Recall that commutators are defined by

\[
T^m_b f(x) = \int (b(x) - b(y))^m K(x - y) f(y) \, dy,
\]

where \( m = 0, 1, 2, \ldots \) and \( K \) is a Calderón–Zygmund kernel. Observe that \( T^0_b \) is the singular integral operator. \( T^1_b = [b, T] \) is the classical commutator introduced by R. Coifman, R. Rochberg and G. Weiss in [CRW] where they showed that it is a bounded operator on \( L^p(\mathbb{R}^n) \), \( 1 < p < \infty \), when \( b \) is a \( BMO \). They also showed that \( BMO \) is also a necessary condition when \( T \) is the vector–valued Riesz transform.

It is shown in [P3] the following pointwise estimate which can be seen as a version of condition (D) for commutators.

**Lemma 6.4.1** Let \( b \in BMO \), and let \( 0 < \delta < \epsilon < 1 \) Then, there exists \( C = C_{\delta, \epsilon} > 0 \) such that

\[
M^\#_{\delta}(T^m_b f)(x) \leq C \sum_{j=0}^{m-1} \|b\|_{BMO}^j M_j(T^j_b f)(x) + \|b\|_{BMO}^m M^{m+1} f(x) \tag{6.12}
\]

for all smooth functions \( f \), where \( M_j(f) = M(|f|^\epsilon)^{1/\epsilon} \).

Using this lemma we can give a version for commutators of Coifman’s theorem 6.4. See [P5].

**Theorem 6.4.2** Let \( 0 < p < \infty \) and let \( w \in A_\infty \) and \( b \in BMO \). Then, there exists a constant \( C \) depending upon the \( A_\infty \) constant of \( w \), such that

\[
\int_{\mathbb{R}^n} |T^m_b f(x)|^p w(x) dx \leq C \|b\|_{BMO}^m \int_{\mathbb{R}^n} M^{m+1} f(x)^p w(x) dx, \tag{6.13}
\]

for any function \( f \) with finite left hand side. Furthermore, the inequality does not hold if \( M^{m+1} \) is replaced by the smaller operator \( M^m \).

As above we can give can give the following version of Boyd’s theorem for commutators.
**Theorem 6.4.3** Let $T^m_b$ be a commutator operator as above with $b \in BMO$. As in Theorem 6.3.1 we let $X$ be a BFS such that the Hardy–Littlewood maximal function is bounded on $X'$. Then

$$\|T^m_b f\|_X \leq c \|M^{m+1} f\|_X.$$  

Therefore, if the Hardy–Littlewood maximal function is also bounded on $X$ then $T^m_b$ is bounded on $X$.

In particular, if $X$ is any Rearrangement Invariant Function Space such that $0 < \alpha_X \leq \pi_X < 1$ we have

$$T^m_b : X \to X.$$  

Another application of this lemma was derived in [P3] where it is shown that the commutators $[b, T]$ are not in general of weak type $(1, 1)$ but they satisfy the following “$L \log L$” type estimate:

**Theorem 6.4.4** Let $b \in BMO$ and let $\Phi_m(t) = t \log^m (e + t)$. Then there exists a constant $C > 0$ depending upon the BMO constant of $b$ such that for all $t > 0$

$$\left| \{ y \in \mathbb{R}^n : |T^m_b f(y)| > t \} \right| \leq C \int_{\mathbb{R}^n} \Phi_m \left( \frac{|f(y)|}{t} \right) dy,$$

(6.14) for all bounded functions $f$ with compact support.

A different proof of this result which allows to include non $A_\infty$ weights was given in [PP]:

$$w\left( \{ x \in \mathbb{R}^n : |[b, T] f(x)| > \lambda \} \right) \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{|f(y)|}{t} \right) M_{L(\log L)^{1+\epsilon}}(w)(x) dx.$$  

(6.15)

Comparing these results with similar estimates for Calderón-Zygmund operators, they show that these commutators carry a higher degree of singularity.

These results have been further exploited in [PT] to the case of “multilinear” commutators of the form

$$T^m_b f(x) = \int_{\mathbb{R}^n} \left[ \prod_{j=1}^m (b_j(x) - b_j(y)) \right] K(x, y) f(y) dy,$$

where $K$ is a Calderón-Zygmund kernel, with vector symbol $\vec{b} = (b_1, \ldots, b_m)$.  

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Chapter 7

Appendix I: The Calderón–Zygmund covering lemma

In this Appendix we recall the definition and basic properties of the (dyadic) Calderón–Zygmund cubes which is also the basis of the Calderón–Zygmund decomposition.

First we recall how to build the family \( D \) of (left open) dyadic cubes on \( \mathbb{R}^n \). For any integer \( k \) we consider the lattice \( \Lambda_k = 2^{-k}\mathbb{Z}^n \). The mesh \( D_k \) is the collection of all left open cubes determined by \( \Lambda_k \). The cubes in \( D_k \) are congruent, each have sides length \( 2^k \) and volume \( 2^{kn} \). The mesh \( D_k \) is a disjoint cover of \( \mathbb{R}^n \). It is also a refinement of \( D_{k-1} \), because every cube \( Q \) in \( D_k \) is contained in exactly one cube \( Q' \in D_{k-1} \) (the "ancestor" of \( Q \)). As a matter of fact \( Q \) is one of the \( 2^n \) congruent cubes obtained by bisecting the sides of \( Q' \in D_{k-1} \).

We will denote by \( D = \bigcup_{k \in \mathbb{Z}} D_k \) as the family of dyadic cubes of \( \mathbb{R}^n \). Dyadic cubes are very useful for constructions of disjoint covers. Here there are some very useful properties of the dyadic cubes which can be verified without any difficulty.

a) Every two dyadic cubes are either disjoint or one of them contains the other.

b) Every family \( \mathcal{F} \subset \mathcal{M} \) of dyadic cubes contains a disjoint subfamily \( \mathcal{F}' \subset \mathcal{F} \) such that

\[
\cup_{Q \in \mathcal{F}} Q = \cup_{Q \in \mathcal{F}'} Q
\]

c) Each point \( x \in \mathbb{R}^n \) belongs to one and only one cube in \( D_k, k \in \mathbb{Z} \). Therefore \( x \) generates a unique two-way infinite chain \( \{Q_k\}_{k=-\infty}^{\infty}, Q_{k+1} \subset Q_k \) such that

\[
\{x\} = \bigcap_{k=0}^{\infty} Q_k.
\]

In a similar way, given a cube \( Q \) we can construct its family of dyadic cubes \( D(Q) \). Indeed, by bisecting each side of the cube \( Q \) we obtain \( 2^n \) "descendent" from \( Q \) which is
the first generation $D_1$; then we keep dividing in this way each descendent. Then $\mathcal{D}(Q) = \bigcup_{k=0}^{\infty} D_k$ is the family of dyadic cubes relative to $Q$. They share the same properties as the dyadic cubes $\mathcal{D}$. In particular each $x \in Q$ generates a unique one-way infinite chain $\{Q_k\}_{k=0}^{\infty}$, $Q_{k+1} \subset Q_k$, such that $\{x\} = \bigcap_{k=0}^{\infty} Q_k$

This construction is intimately to dyadic maximal function. In the local case is the defined by

$$M_Q^d f(x) = M_Q f(x) = \sup_{x \in P} \frac{1}{|P|} \int_P |f(y)| dy.$$ 

The global dyadic maximal function can be defined in a very similar way by replacing $\mathcal{D}(Q)$ by $\mathcal{D}$.

For a nonnegative function $f$ in $L^1(Q)$ we define

$$\Omega_t = \{x \in Q : M_Q f(x) > t\}.$$

The we have the following local version of the Calderón–Zygmund covering lemma.

**Lemma 7.0.5 (local Calderón–Zygmund covering lemma)** Let $f$ be a nonnegative function integrable on a cube $Q$ and let $t \geq \frac{1}{|Q|} \int_Q f$. Then, if $\Omega_t$ is not empty, there is a family of pairwise disjoint dyadic cubes $\{Q_i\}$ (the Calderón–Zygmund cubes of $f$ at level $t$) such that $\Omega_t = \bigcup_i Q_i$,

with

$$t < \frac{1}{|Q_i|} \int_{Q_i} f(x) dx \leq 2^n t$$

(7.1) for each $i$. As a consequence we have the weak type $(1,1)$ property with constant 1:

$$|\Omega_t| \leq \frac{1}{t} \int_Q f(x) dx.$$

(7.2)

We also have that

$$f(x) \leq t \quad \text{for a.e. } x \notin \bigcup_i Q_i$$

(7.3)

and we can further refine (7.2) with

$$\frac{1}{2^n} \int_{x \in Q : f(x) > t} f(x) dx \leq t|\Omega_t| \leq \int_{x \in Q : f(x) > t/2} f(x) dx.$$

(7.4)

Inequality (7.4) seems to be due to Wiener.

**Proof:** For each $x \in \Omega_t$, there is a dyadic cube $P$ containing $x$ and such that

$$\frac{1}{|P|} \int_P f(y) dy > t.$$
Take the (unique) maximal (under inclusion) of these $P$'s and denote it by $P^x$. Observe that $P^x$ is strictly contained in $Q$ since we assume $t \geq \frac{1}{|Q|} \int_Q f$. We let $\{Q_i\}$ denote this collection of maximal dyadic cubes which are pairwise disjoint. Then it is clear that

$$\Omega_t = \bigcup_i Q_i,$$

as well as the left hand side of (7.1). The right hand side of (7.1) follows by the maximality of each of the cubes $\{Q_i\}$ and by the doubling property of the Lebesgue measure. The weak type $(1, 1)$ property follows immediately from the left hand side of (7.1).

The weak type $(1, 1)$ property together with a standard method (see [GCRdF] p. 139) implies the dyadic version of the Lebesgue differentiation theorem, namely that a.e. $x \in Q$

$$f(x) = \lim_{k \to \infty} \frac{1}{|Q_k|} \int_{Q_k} f(y) dy,$$

where $\{Q_k\}_{k=0}^\infty$ is the unique chain of dyadic cubes such that $\{x\} = \bigcap_{k=0}^\infty Q_k$. Then $f(x) \leq M_Q f(x)$ and therefore $f(x) \leq t$ if $x \notin \bigcup_i Q_i = \Omega_t$.

To prove the left hand side of (7.4) we use that by the Lebesgue differentiation theorem we have $f(x) \leq M f(x)$ and

$$\int_{\{x \in Q : f(x) > t\}} f(x) dx = f(\{x : f(x) > t\}) \leq f(\Omega_t) = f(\bigcup_i Q_i) = \sum_i f(Q_i) \leq 2^n t \sum_i |Q_i| \leq 2^n t |\Omega_t|.$$

To prove the right hand side we use that $M$ is bounded on $L^\infty$. Indeed, we write $f$ as $f = f_1 + f_2$, where $f_1(x) = f(x)$ if $f(x) > \frac{t}{2}$, and $f_1(x) = 0$ otherwise. Then $M f(x) \leq M f_1(x) + M f_2(x) \leq M f_1(x) + \frac{t}{2}$. and it follows immediately that

$$|\Omega_t| \leq \int_{x : f(x) > t/2} \frac{f(x)}{t} dx$$

by the weak type $(1, 1)$ estimate.

\[\square\]

There is a global version of this Calderón–Zygmund covering lemma. The minimal assumption on $f$ to construct the Calderón–Zygmund cubes is that it must belong to the following class of functions:

$$CZ = \{f \in L^1_{loc}(R^n) : \frac{1}{|Q|} \int_Q f \to 0 \text{ when } Q \uparrow R^n\}.$$ 

Observe that $\bigcup_{p \geq 1} L^p \subset CZ$, in fact there are functions $f \in CZ \setminus \bigcup_{p \geq 1} L^p$. 

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Then if $f \in CZ$ and $t > 0$ (compare with the local situation where we must assume $t \geq \frac{1}{|Q|} \int_Q f$) there is a family of pairwise disjoint dyadic cubes $\{Q_i\}$ verifying all the properties listed in the local Calderón–Zygmund covering lemma where this time $\Omega_t = \{x \in Q : Mf(x) > t\}$ and

$$M^d f(x) = M f(x) = \sup_{P \in D} \frac{1}{|P|} \int_P |f(y)| \, dy.$$  

(7.5)
Chapter 8

Appendix II: The $A_\infty$ condition of Muckenhoupt

Recall that the $A_\infty$ class of weights is defined naturally by

$$ A_\infty = \cup_{p>1} A_p, $$

since the $A_p$ class of weights are increasing on $p$. This condition was also introduced by Muckenhoupt in [Mu] and was further studied in the classical paper by R. Coifman and C. Fefferman [CF]. We have the following characterization; and specially condition f) is specially interesting in applications.

Lemma 8.0.6 For a weight $w$ the following conditions are equivalent:

a) $w \in A_\infty$;

b) there exists a positive constant $C$ such that

$$ \frac{1}{|Q|} \int_Q w \, dx \leq C \exp \left( \frac{1}{|Q|} \int_Q \log w \, dx \right) $$

(8.1)

c) there are positive constants $\alpha$ and $\beta$ such that

$$ |\{x \in Q : w(x) \leq \beta w_Q\}| \leq \alpha |Q| $$

(8.2)

d) There exists a constant $C$ such that for $\lambda > w_Q$

$$ w(\{x \in Q : w(x) > \lambda\}) \leq C \lambda |\{x \in Q : w(x) > \beta \lambda\}|. $$

(8.3)

e) $w$ satisfies a reverse Hölder inequality, namely there are positive constants $c$ and $\delta$ such that

$$ \left( \frac{1}{|Q|} \int_Q w^{1+\delta} \, dx \right)^{\frac{1}{\delta}} \leq \frac{c}{|Q|} \int_Q w \, dx $$

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f) there are positive constants \( c \) and \( \rho \) such that for any cube \( Q \) and any measurable set \( E \) contained in \( Q \) then
\[
\frac{w(E)}{w(Q)} \leq c\left(\frac{|E|}{|Q|}\right)^\rho. \tag{8.4}
\]

g) \( w \) satisfies the following condition: there are positive constants \( \alpha, \beta < 1 \) such that whenever \( E \) is a measurable set of a cube \( Q \)
\[
\frac{|E|}{|Q|} < \alpha \quad \text{implies} \quad \frac{w(E)}{w(Q)} < \beta. \tag{8.5}
\]

**Proof:** Here we combine the methods from [CF] and [GCRdF].

\( a) \Rightarrow b) \)

By Jensen’s inequality it is enough to show that
\[
\frac{1}{|Q|} \int_Q w \, dx \leq C \exp\left(\frac{1}{|Q|} \int_Q \log w \, dx\right).
\]

Since the \( A_p \) classes are increasing on \( p \) if \( w \in \bigcup_{p>1} A_p \) there exists some \( p_0 > 1 \) such that \( w \in A_p \) for \( p \geq p_0 \). Then, there exists a constant \( K \) such that for \( p \geq p_0 \)
\[
\left(\frac{1}{|Q|} \int_Q w \, dx\right)^p \left(\frac{1}{|Q|} \int_Q w^{1-p'} \, dx\right)^{p-1} \leq K.
\]

Letting \( p \) tend to \( \infty \) we obtain \( b) \).

\( b) \Rightarrow c) \)

Dividing \( w \) by an appropriate constant (to be precise \( \exp\left(\frac{1}{|Q|} \int_Q \log w \, dx\right) \)) we may assume that \( \int_Q \log w \, dx = 0 \) and, consequently the hypothesis (8.1) becomes \( w_Q \leq C \).
\[
|\{x \in Q : w(x) \leq \beta w_Q\}| \leq |\{x \in Q : w(x) \leq \beta C\}|
\]
\[
= |\{x \in Q : \log(1 + \frac{1}{w(x)}) \geq \log(1 + \frac{1}{\beta C})\}|
\]
\[
\leq \frac{1}{\log(1 + \frac{1}{\beta C})} \int_Q \log(1 + \frac{1}{w}) \, dx = \frac{1}{\log(1 + \frac{1}{\beta C})} \int_Q \log(1 + w) \, dx
\]
since \( \int_Q \log w \, dx = 0 \). Now, since \( \log(1 + t) \leq t, t \geq 0 \), we get:
\[
|\{x \in Q : w(x) \leq \beta w_Q\}| \leq \frac{1}{\log(1 + \frac{1}{\beta C})} \int_Q w \, dx \leq \frac{C|Q|}{\log(1 + \frac{1}{\beta C})} \leq \frac{1}{2} |Q|,
\]
if we choose \( \beta \) small enough.

\( c) \Rightarrow d) \)
Since we assume that $\lambda > w_Q$ we may consider the local Calderón–Zygmund covering lemma of $w$ (see Appendix I) and we find a family of disjoint cubes $\{Q_j\}$ satisfying
\[
\lambda < \frac{1}{|Q_j|} \int_{Q_j} w \, dx \leq 2^n \lambda
\]
for each $j$. By the properties of the cubes combined with (8.2) we have
\[
w(\{x \in Q : w(x) > \lambda\}) \leq \sum_j w(\{x \in Q : w(x) > \lambda\}) \leq 2\lambda \sum_j |Q_j| \leq 2\lambda \sum_j \int_{Q_j} w \, dx \leq \frac{2\lambda}{1 - \alpha} |\{x \in Q : w(x) > \beta w\}|
\]
since $w_{Q_j} > \lambda > w_Q$.

d) $\Rightarrow$ e)

We will be using the formula
\[
\int_X f(x)^p \, d\nu = p \int_0^\infty \lambda^p \nu(\{x \in X : f(x) > \lambda\}) \frac{d\lambda}{\lambda}
\]
which holds for every nonnegative measurable function $f$ and in any arbitrary measure space $(X, \nu)$ with nonnegative measure $\nu$. Then for arbitrary positive $\delta$ we have
\[
\frac{1}{|Q|} \int_Q w^{1+\delta} \, dx = \frac{\delta}{|Q|} \int_0^\infty \lambda^\delta w(\{x \in Q : w(x) > \lambda\}) \frac{d\lambda}{\lambda}
\]
\[
= \frac{\delta}{|Q|} \int_{w_Q} w^\delta \{x \in Q : w(x) > \lambda\} \frac{d\lambda}{\lambda} + \frac{\delta}{|Q|} \int_{w_Q} \lambda^{\delta+1} \{x \in Q : w(x) > \beta \lambda\} \frac{d\lambda}{\lambda}
\]
\[
\leq (w_Q)^{1+\delta} + \frac{\delta}{|Q|} \int_{w_Q} \lambda^{\delta+1} |\{x \in Q : w(x) > \beta \lambda\}| \frac{d\lambda}{\lambda}
\]
\[
\leq (w_Q)^{1+\delta} + \frac{C \delta}{|Q|} \int_{w_Q} \lambda^{\delta+1} \frac{1}{\beta^{1+\delta}} |\{x \in Q : w(x) > \beta \lambda\}| \frac{d\lambda}{\lambda}
\]
\[
\leq (w_Q)^{1+\delta} + \frac{C \delta}{|Q|} \int_{Q} w^{1+\delta} \, dx.
\]
If we choose $\delta$ small enough such that $\frac{C \delta}{\beta^{1+\delta}} < 1$ the last term can be absorbed by the first term of the string of inequalities.

e) $\Rightarrow$ f)
This is just Hölder inequality with \( r = 1 + \delta \). Indeed if \( E \subset Q \)

\[
\frac{w(E)}{|Q|} \leq \frac{1}{|Q|} \int_Q \chi_E w \, dx \leq \left( \frac{1}{|Q|} \int_Q w^r \, dx \right)^{1/r} \left( \frac{|E|}{|Q|} \right)^{1/r'}
\]

and this implies the \( A_\infty \) condition (8.4) with \( \rho = 1/r' \).

\( f) \Rightarrow g) \)

This is immediate.

\( g) \Rightarrow c) \)

First observe that condition (8.5) is equivalent to saying that there are positive constants \( \alpha, \beta < 1 \) such that whenever \( E \) is a measurable set of a cube \( Q \)

\[
\frac{w(E)}{w(Q)} < \alpha \quad \text{implies} \quad \frac{|E|}{|Q|} < \beta.
\]

(8.6)

Then let \( E = \{ x \in Q : w(x) > bw_Q \} \) where \( b \in (0, 1) \) is going to be chosen now and let \( E' = Q \setminus E = \{ x \in Q : w(x) \leq bw_Q \} \). Then \( w(E') \leq bw_Q |E'| \leq bw(Q) \). Then if we take \( b = \beta \) we have that \( |E'| \leq \alpha |Q| \) and hence \( |E| \geq (1 - \alpha) |Q| \). This yields (8.2).

Therefore we have shown that \( c) \iff d) \iff e) \iff f) \iff g) \)

\( c) \Rightarrow a) \)

We use again that condition e) is symmetric, namely that condition (8.6) holds. We also use that the measure \( w \, dx \) is doubling. Now, if we write \( dx = w^{-1} w \, dx \) and since \( c) \iff e) \)

we have that there are positive constants \( c \) and \( \delta \) such that

\[
\left( \frac{1}{w(Q)} \int_Q (w^{-1})^{1+\delta} w \, dx \right)^{1/(1+\delta)} \leq \frac{c}{w(Q)} \int_Q w^{-1} w \, dx.
\]

Hence

\[
\frac{w(Q)}{|Q|} \left( \frac{1}{|Q|} \int_Q w^{-\delta} \, dx \right)^{1/\delta} \leq C.
\]

Then, if we let \( \delta = \frac{1}{p-1} \), that is, \( p = \frac{1}{\delta} + 1 > 1 \) we have that \( w \in A_p \).

The proof of the Lemma is now complete.

\[ \square \]

It is now very easy to see the following important openness property shared by the \( A_p \) classes.
Corollary 8.0.7 Let $p > 1$, then
\[ w \in A_p \text{ implies } w \in A_{p-\epsilon}, \]
for some $\epsilon$ small enough depending upon the $A_p$ constant.
Chapter 9

Appendix III: Exponential self-improving property

- Generalized John-Nirenberg Inequalities

In the first part of this appendix we will prove the John–Nirenberg theorem in a bit of more generality. Our starting point is

\[
\frac{1}{|Q|} \int_Q |f(y) - f_Q| dy \leq a(Q),
\]

where we assume that the functional \( a \) is nondecreasing, namely if \( P \subset Q \) \( a(P) \leq a(Q) \).

We recall the normalized norm

\[
\|g\|_{expL(Q,w)} = \inf\{\lambda > 0 : \frac{1}{w(Q)} \int_Q \left( \exp\left(\frac{|f(y)|}{\lambda}\right) - 1 \right) w(y) dy \leq 1\}.
\]

**Theorem 9.0.8** Let \( f \) be a locally integrable function such that

\[
\inf_{\alpha} \left( (f - \alpha) \chi_Q \right)^* (\lambda|Q|) \leq c_\lambda a(Q). \tag{9.1}
\]

where \( a \) is nondecreasing. Let \( w \in A_\infty \). Then, there exists a constant \( C \) such that for all the cubes \( Q \) in \( \mathbb{R}^n \)

\[
\|f - f_Q\|_{exp(L)(Q,w)} \leq C a(Q). \tag{9.2}
\]

As a sample consider in the one dimensional case the generalized Poincaré inequality

\[
\frac{1}{|I|} \int_I |f(y) - f_I| dy \leq \int_I g(y) dy \tag{9.3}
\]

for some \( g \) locally integrable, then \( f \) has an exponential decay, even more we have the uniform decay.
\[ \|f - f_t\|_{\text{exp}(L;I,w)} \leq C \int_I g(y) \, dy. \]

**Remark 9.0.9** We want to remark it is possible to show that this generalized John-Nirenberg can be seen as a limit of certain appropriate generalized Trudinger's inequality. For this point of view see [MP2] or below.

**Proof of the Theorem:**

By Theorem (3.0.3) and since \( a : Q \rightarrow (0, \infty) \) is nondecreasing it is easy to see that we can assume that \( f \) is locally integrable and that for some constant \( c \)

\[ \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy \leq c a(Q) \tag{9.4} \]

for all the cubes \( Q \). It is enough to show that there is a constant \( C \) such that for each cube \( Q \)

\[ \frac{1}{|Q|} \int_Q \exp \left( \frac{|f(y) - f_Q|}{C a(Q)} \right) \, dy \leq 1. \]

which will follow by standard arguments from the inequality

\[ \frac{e^{\beta t}}{w(Q)} w \left( \{ x \in B : \frac{|f(x) - f_Q|}{a(Q)} > t \} \right) \leq 1. \]

where \( \beta \) is an appropriate constant independent of \( t > 0 \).

It is clear that we may assume that \( t > t_0 \) for a large enough universal constant \( t_0 \).

For each \( t > 0 \) we let \( E(Q, t) = \{ x \in Q : |f(x) - f_Q| > t a(Q) \} \) and

\[ \varphi(t) = \sup_{Q \in Q} \frac{w(E(Q,t))}{w(Q)}, \]

By hypothesis \( \varphi(t) \leq \min \{ \frac{1}{t}, 1 \} \).

Recall since \( w \) is a \( A_\infty \) weight there are positive constants \( c \) and \( \rho \) such that for any cube \( Q \) and any measurable set \( E \) contained in \( Q \) then

\[ \frac{w(E)}{w(Q)} \leq c \left( \frac{|E|}{|Q|} \right)^\rho. \]

**Lemma 9.0.10** There exists \( \gamma > 1 \) such that for every \( t > \gamma 2^n \)

\[ \varphi(t) \leq \frac{1}{e} \varphi(t - \gamma 2^n). \tag{9.5} \]

Reiterating this inequality \( \left[ \frac{t}{\gamma 2^n} \right] \) times we get for \( t > \gamma 2^n \)

\[ \varphi(t) \leq C e^{-ct} \]
We can consider the local Calderón–Zygmund covering lemma (see Appendix I) of \( f - f_Q \) relative to \( Q \) at level \( \gamma > 1 \), since

\[
\frac{1}{|Q|} \int_Q \frac{|f(y) - f_Q|}{C a(Q)} \, dy \leq 1 < \gamma.
\]

This yields a collection of dyadic subcubes of \( Q \), \( \{Q_i\} \), maximal with respect to inclusion, satisfying

\[
\gamma < \frac{1}{|Q_i|} \int_{Q_i} \frac{|f(y) - f_Q|}{a(Q)} \, dy \leq \gamma 2^n\tag{9.6}
\]

for each integer \( i \). Also by the Lebesgue differentiation theorem

\[
E(Q, \gamma) \subset \bigcup_i Q_i.
\]

Now, for \( t > \gamma 2^n \) and using the monotonicity of \( a \) together with

\[
|f(x) - f_Q| \leq |f(x) - f_{Q_i}| + |f_{Q_i} - f_Q| \leq |f(x) - f_{Q_i}| + \gamma 2^n a(Q)
\]

we have

\[
w(E(Q, t)) = w(E(Q, \gamma) \cap E(Q, t)) = \sum_i w(\{x \in Q_i : |f(x) - f_Q| > a(Q) \, t\})
\]

\[
\leq \sum_i w(\{x \in Q_i : |f(x) - f_{Q_i}| > a(Q_i) \, (t - \gamma 2^n)\})
\]

\[
\leq \sum_i w(Q_i) \varphi(t - \gamma 2^n) = \varphi(t - \gamma 2^n) w(\bigcup_i Q_i).
\]

Now by the \( A_\infty \) condition

\[
w(\bigcup_i Q_i) \leq c \left( \frac{|\bigcup_i Q_i|}{|Q|} \right)^\delta w(Q),
\]

but by (9.6) and by disjointness of the cubes

\[
|\bigcup_i Q_i| = \sum_i |Q_i| \leq \frac{1}{\gamma} \int_Q \frac{|f(y) - f_Q|}{a(Q)} \, dy \leq \frac{1}{\gamma} |Q|
\]

and hence we conclude for \( t > \gamma 2^n \):

\[
\varphi(t) \leq \frac{c}{\gamma \delta} \varphi(t - \gamma 2^n).
\]

To conclude it is enough to choose \( \gamma \) such that \( \frac{c}{\gamma \delta} < \frac{1}{e} \).
Generalized Trudinger Inequalities

Recall that the Sobolev Embedding Theorem says that functions in $W^{1,q}_{\text{loc}}(\mathbb{R}^n)$ actually lie in $L^r_{\text{loc}}$ for $r = nq/(n-q)$ when $1 \leq q < n$. We have already mentioned that a more precise version of this statement is the following local inequality:

$$\left( \frac{1}{|Q|} \int_Q |f - f_Q|^r \right)^{1/r} \leq C \ell(Q) \left( \frac{1}{|Q|} \int_Q |\nabla f|^n \right)^{1/n} \tag{9.7}$$

for any cube $Q$. When $q$ tends to $n$ the constant $C$ on the righthand side blows up and so the limiting case with $q = n$, i.e., $W^{1,n}_{\text{loc}} \subseteq L^\infty_{\text{loc}}$, is false. The correct result in this instance is that $W^{1,n}_{\text{loc}}$ lies locally in the class $\text{exp} L^{n'}$. The corresponding inequality, called Trudinger’s Inequality, is

$$\|f - f_Q\|_{\text{exp} L^{n'}(Q)} \leq C \left( \int_Q |\nabla f|^n \right)^{1/n}. \tag{9.8}$$

It should be mentioned that it was also derived by Yudovich in [Y]. See [GT], for instance, for a proof in the case that $f$ has compact support. The norm on the left hand side is the Luxemburg (or Orlicz) norm associated to the function $\Phi(t) = \exp t^{n'} - 1$ (see Section 4.1). Trudinger’s inequality has proved to be one of the key results in the study of parabolic and elliptic equations at the critical index (which is $n$ in the case of $\mathbb{R}^n$). In addition, the sharp value for the constant $C$ appearing on the righthand side (see Moser [Mo], and Adams [Ad] for the higher dimensional version) plays a major role in geometry, especially in the problem of prescribing Gaussian curvatures on spheres.

Motivated by the work [HaK2] we consider in [MP2] the following more general situation. As above we will consider locally integrable functions such that for every cube $Q$

$$\frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy \leq a(Q),$$

where $a : Q \to [0, \infty)$. The question is: What kind of geometric condition, in the spirit of (2.15), do we have to impose to get an exponential estimate similar to (9.8)?

**Definition 9.0.11.** Let $1 < r < \infty$. We say that the functional $a$ satisfies the $T_r$ condition if there exists a finite constant $c$ such that for each cube $Q$ and any family $\Delta$ of pairwise disjoint subcubes of $Q$,

$$\sum_{P \in \Delta} a(P)^r \leq C^n\ a(Q)^r. \tag{9.9}$$

We will use $\|a\|$ to denote the smallest constant $C$ for which (9.9) holds. We always have $\|a\| \geq 1$. The main examples are $a(Q) = \left( \int_Q g^r \right)^{1/r}$ where $g \in L^r_{\text{loc}}(\mathbb{R}^n)$ and, more generally,

$$a(Q) = \nu(Q)^{1/r},$$

where $\nu$ is a measure on $\mathbb{R}^n$.
where \( \nu \) is a locally finite measure. Observe that these conditions are increasing, in the sense that \( r < s \Rightarrow T_r \subset T_s \). Also observe that this condition is much stronger than the \( D_r \) condition (2.15) since in particular \( a \) is increasing.

**Theorem 9.0.12** Assume that the functional \( a \) satisfies the \( T_r \) condition for some \( 1 < r < \infty \), and that \( w \) is a doubling weight. Let \( f \) is a locally integrable function such that for all cubes \( Q \) in \( \mathbb{R}^n \)

\[
\inf_{\alpha} \left( (f - \alpha) \chi_Q \right)^* (\lambda|Q|) \leq c_\lambda a(Q). \tag{9.10}
\]

Then there exists a constant \( C \) independent of \( f \) such that

\[
\|f - f_Q\|_{\exp L^{r'}(Q,w)} \leq C \ a(Q). \tag{9.11}
\]

for any cubes \( Q \).

Note the important fact that the functional \( a \) need not depend on \( f \), in contrast to the classical case where the functional is given by \( a(Q) = \left( \int_Q |\nabla f|^n \right)^{1/n} \) As a matter of fact we could work in spaces with no differential structure at all. See [MP2].

**Remark 9.0.13** A point of interest here is that the limiting “\( r = \infty \)” case of this Theorem is just above generalized John–Nirenberg theorem 9.0.8. Indeed, let \( T_\infty \) be defined as those functionals a “essentially” in the sense that for some finite constant \( c \) and for arbitrary cubes \( Q, P \) with \( P \subset Q \) then \( a(P) \leq c \ a(Q) \) (Theorem 9.0.8 also holds in this case). Observe that \( T_r \subset T_\infty \) and that in fact the condition \( T_\infty \) is simply the limit as \( r \to \infty \) of the condition \( T_r \). Recall that the functional \( a(Q) \equiv 1 \) is the functional associated to the space \( BMO \), satisfies \( T_\infty \).

**Proof:** To simplify the proof we just consider the unweighted case \( w \equiv 1 \).

As above, by Theorem (3.0.3) and since \( a: Q \to (0, \infty) \) is in particular nondecreasing it is easy to see that we can assume that \( f \) is locally integrable and that for some constant \( c \)

\[
\frac{1}{|Q|} \int_Q |f(y) - f_Q| \ dy \leq c \ a(Q) \tag{9.12}
\]

for all the cubes \( Q \).

We claim that for a fixed cube \( Q \) and for a.e. \( x \in Q \)

\[
|f(x) - f_Q| \leq C \ a(Q) \left( (\log A(x))^{1/r'} + 1 \right). \tag{9.13}
\]

Indeed as in the proof of Lemma 2.3.1, if \( x \in Q \) consider the unique one-way infinite chain of dyadic cubes starting from \( Q = Q_0, \{Q_k\}_{k=0}^\infty \) such that

\[
\{x\} = \cap_{k=-\infty}^\infty Q_k.
\]
See Appendix I. For each \( k \) we let \( f_k = \frac{1}{|Q_k|} \int_{Q_k} f \). Then by the Lebesgue differentiation theorem we have a.e. \( x \in Q \) that

\[
|f(x) - f_Q| = |\lim_{k \to \infty} f_k(x) - f_0| = |f_0 - f_N + \sum_{k=N+1}^\infty (f_{k-1} - f_k)|
\]

\[
\leq |f_0 - f_N| + \sum_{k=N}^\infty |f_k - f_{k+1}| = I_N + II_N
\]

To estimate \( II_N \) we define the following (normalized) maximal function associated to \( a \) by

\[
A(x) = A_Q(x) = \sup_{P \in \mathcal{D}(Q)} \frac{a(P)}{|Q|^{1/r}} \frac{|Q|^{1/r}}{|P|^{1/r}} a(Q).
\]

Then we can estimate \( II_N \) as follows

\[
II_N \leq \sum_{k=N}^\infty \frac{1}{|Q_{k+1}|} \int_{Q_{k+1}} |f - f_{Q_k}| \leq 2^n \sum_{k=N}^\infty a(Q_k)
\]

\[
\leq 2^n A(x) \frac{a(Q)}{|Q|^{1/r}} \sum_{k=N}^\infty |Q_k|^{1/r} \approx A(x) a(Q) 2^{-Nn/r}.
\]

To estimate \( I_N \) we are going to link the cubes \( Q_N \) and \( Q \) by means of a special chain of disjoint cubes \( \{P_i\}_{i=0, \ldots, M} \) contained in \( Q \) where \( P_0 = Q_N \) (in the picture is the smallest nondash cube). We also define \( P_{M+1} = Q \).

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In the picture below, \( \epsilon = 2^{-N} \ell(Q) \) denotes the sidelenght of the first cube. First, we select the face of \( Q \) whose distance to \( Q_N \) is furthest and build a “stairs” of cubes following this direction. To define \( P_1 \) we take the adjacent cube to \( P_0 \) with sidelenght \( 2\ell(P_0) = 2\epsilon \) as in the picture (again is nondash cube). \( P_2 \) is defined in a similar way as shown in the picture. We continue in this way until we reach \( Q_M \) with \( Q_M \subset Q \) and with \( M \approx N \).

\[\]
Now recalling the notation $f_E = \frac{1}{|E|} \int_E f$, we have

$$I_N = |f_{Q_N} - f_Q| \leq |f_Q - f_{Q_M}| + |f_{Q_M} - f_{Q_N}|$$

$$\leq \frac{1}{|Q_M|} \int_{Q_M} |f - f_Q| + \sum_{i=1}^M |f_{P_i} - f_{P_{i+1}}|.$$

The first term is immediate:

$$\frac{1}{|Q_M|} \int_{Q_M} |f - f_Q| \leq \frac{C}{|Q|} \int_{Q} |f - f_Q| \leq C a(Q).$$

As for the second, for each $i = 1, \cdots, M$ we compare $P_i$ and $P_{i+1}$ with a third cube $\tilde{P}_i$ which can be defined as the minimal cube containing $P_i$ and $P_{i+1}$ following the "direction" of the "stairs" (in the picture these are the dash cubes). Then

$$|f_{P_i} - f_{P_{i+1}}| \leq |f_{P_i} - f_{P_i}| + |f_{P_i} - f_{P_{i+1}}| \leq C a(\tilde{P}_i)$$

since $\ell(\tilde{P}_i) \approx \ell(P_i) \approx \ell(P_{i+1})$. Hence, combining these estimates we have

$$I_N \leq C a(Q) + \sum_{i=1}^M a(\tilde{P}_i) \leq C a(Q) + \left( \sum_{i=1}^M a(\tilde{P}_i)^r \right)^{1/r} M^{1/r'}$$

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Now, if we want to use the $T_r$ condition we must have disjoint cubes. However, observe from the picture that the overlap is finite, namely two. Then we can split the family $\{\tilde{P}_i\}$ in two (the odd and even index for instance) and each family is formed by pairwise disjoint cubes. Then

$$I_N \leq C a(Q) + C M^{1/r'} a(Q) \approx N^{1/r'} a(Q).$$

Combining the estimates for $I_N$ and $II_N$ we have

$$|f(x) - f_Q| \leq C a(Q) \left( N^{1/r'} + A(x) 2^{-N/r} \right).$$

We optimize $N$ by choosing $N = N(x)$ to be an integer approximately equal to $\log A(x)$. Then

$$|f(x) - f_Q| \leq C a(Q) \left( (\log A(x))^{1/r'} + 1 \right)$$

and the claim (9.13) follows.

Now if we let $s > 0$, then a simple manipulation of (9.13) yields

$$\exp \left( s \left( \frac{|f(x) - f_Q|}{C a(Q)} \right)^{r'} \right) \leq \exp \left( s (1 + \log A(x)) \right) \leq CA(x)^s$$

for a.e. $x \in Q$. Averaging over $Q$ we have

$$\frac{1}{|Q|} \int_Q \exp \left( s \left( \frac{|f(x) - f_Q|}{C a(Q)} \right)^{r'} \right) dx \leq \frac{C}{|Q|} \int_Q A(x)^s dx.$$

Now if we choose $s < r$ and apply again Kolmogorov’s inequality 2.25 we have

$$\frac{1}{|Q|} \int_Q \exp \left( s \left( \frac{|f(x) - f_Q|}{C a(Q)} \right)^{r'} \right) dx \leq C \|A\|_{L^{r,\infty}(Q)}^s$$

To conclude with the proof of the Theorem we are left with showing the following lemma.

**Lemma 9.0.14**

$$\|A\|_{L^{r,\infty}(Q)} \leq C \|a\|$$

**Proof:** The proof of

$$|\{x \in Q : A(x) > \lambda\}| \leq \frac{C \|a\|^{1/r}}{\lambda^{1/r}} |Q|$$

is by a standard (dyadic type) covering lemma. If $A(x) > \lambda$ then we can take the maximal dyadic subcube $P$ of $Q$ with $x \in P$ such that

$$\frac{a(P)}{|P|^{1/r}} \frac{|Q|^{1/r}}{a(Q)} > \lambda.$$
Denote this family by \{P_i\} which by maximality they are disjoint. Since each \(P_i\) satisfies last inequality we raise it to the power \(r\) to get

\[
\left| \{x \in Q : A(x) > \lambda \} \right| = \sum_i |P_i| \leq \frac{|Q|}{\lambda^r a(Q)^r} \sum_i |P_i| \frac{a(P_i)^r}{|P_i|} \leq \frac{1}{\lambda^r} \|a\| \|Q\|
\]

by the \(T_r\) condition. This means that \(\|A\|_{L^r(Q)} \leq \|a\|\).

\(\square\)

The proof of the Theorem is now complete.

\(\square\)
Bibliography


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